Applications of Ultraprime Banach Algebras in the Theory of Elementary Operators

Dissertation
der Mathematischen Fakultät
der Eberhard-Karls-Universität
zu Tübingen

zur Erlangung des Grades eines
Doktors der Naturwissenschaften

vorgelegt von

Martin Mathieu

aus
Heidelberg

1986

OBSV.
LINCOLN
VOL.

Digitized by the Internet Archive
in 2026

https://archive.org/details/IA41543903_0049

Applications of Ultraprime Banach Algebras in the Theory of Elementary Operators

Dissertation
der Mathematischen Fakultät
der Eberhard-Karls-Universität
zu Tübingen

zur Erlangung des Grades eines
Doktors der Naturwissenschaften

vorgelegt von

Martin Mathieu

aus
Heidelberg

1986

Tag der mündlichen Qualifikation: *27. Oktober 1986*

Dekan: *Prof. Dr. H. Heyer*
1. Berichterstatter: *Prof. Dr. R. Nagel*
2. Berichterstatter: *Prof. Dr. E. Christensen (Kopenhagen)*

This thesis is dedicated to Susanne and Janine
— without them it would not exist.

As soon as you have become a master
of one field,
you should become a student of another.

from the Chinese

PREFACE

The present thesis is an attempt to develop a theory of elementary operators on Banach algebras on the basis of one general principle and to derive from this characterizations of properties of elementary operators like invertibility, compactness, or (complete) positivity.

Elementary operators have been studied intensively during the past years. Very different questions were tackled and solved by equally different techniques. As a consequence, the results were obtained only under very restricted assumptions, e.g. only for the algebra $\mathcal{L}(H)$ of all bounded operators on some Hilbert space. In many cases it is not clear to which extent they can be generalized. On the other hand, elementary operators have found various applications (some will be indicated in the Introduction and in the subsequent chapters). It therefore appears desirable to have a more handy theory.

The idea on which our work is based is very simple. An elementary operator is a linear combination of (two-sided) multiplications, and the main obstacle which prevents from a direct conclusion of properties of a multiplication $M_{a,b}$ to properties of the multiplying elements a and b is that, in general, $M_{a,b}$ can be zero without a or b being zero. This obstacle can be removed by merely considering *prime algebras*. Then, we can hope for a simpler theory. This hope proves to be well-founded for C*-algebras, and also for Banach algebras if we claim in addition that some ultrapower of the algebra is

prime. This leads to the concept of *ultraprime algebras*.

Therefore this thesis is dominated by the interplay between the ideal structure of a Banach algebra and the elementary operators defined on it. The organization is as follows. Chapters 1 through 3 contain a discussion of ultraprime algebras in view of the later applications to elementary operators which will be carried out in Chapters 4 through 7. Two Appendices provide the prerequisites on ultrapowers (necessary in Chapter 1) and on joint spectra (necessary in Chapter 7). All this is prepared by the Introduction, and completed by the Bibliography and an Index of Symbols.

This thesis would not exist without the help of many. I owe gratitude to my friends, teachers, and colleagues at the Universities of Cork, Dublin, Ljubljana, Santa Barbara, and Tübingen for all their support. I am particularly indebted to Professors C.A. Akemann and T.T. West for invitations to the University of California at Santa Barbara and to Trinity College, Dublin, respectively. I am grateful to Dr. B. Kümmerer for a clue both in Theorem 5.5 and 5.7, and to my advisor Prof. Dr. R. Nagel for his steady encouragement during the preparation of the thesis. I also gratefully acknowledge the support by the Studienstiftung des deutschen Volkes and by the DAAD. Thanks to all of them.

Tübingen, Summer 1986 M.M.

CONTENTS

Introduction 1

1. The concept of ultraprime Banach algebras 9

2. The Banach algebra of quotients of an ultraprime Banach algebra 18

3. Basic properties of prime C*-algebras 32

4. The identity S = 0 42

5. Positivity of elementary operators 59

6. Elementary operators which are compact or weakly compact 71

7. The spectra of elementary operators 89

Appendix 1. Background on ultraproducts 107

Appendix 2. Background on joint spectra 118

Bibliography 127

Index of Symbols 137

INTRODUCTION

The theory of elementary operators has a long history going back to at least the beginning of this century. In 1900 a result by Stéphanos was published ([127]) which stated that the eigenvalues of the multiplication $M_{a,b}x = axb$, where a, b and x are complex $n\times n$-matrices, are given by the products of the eigenvalues of a and b. Matrix equations of the form $\sum A_k X B_k = C$ had already been discussed before by Sylvester (cf. [104], chap. VIII). A number of years later, a systematic investigation of the spectra of elementary operators $Sx = \sum_{j=1}^{n} a_j x b_j$ was initiated by Lumer and Rosenblum [102] in 1959. They gave a description of the spectrum of S in the case when a_j and b_j are images under holomorphic functions of bounded operators a and b on a Banach space E (ibid., thm 10). Their main tool was the following property of the algebra $L(E)$.

(*) *For any two sequences $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ of operators with norm one there is a bounded sequence $(z_k)_{k\in\mathbb{N}}$ in $L(E)$ such that the sequence $(x_k z_k y_k)_{k\in\mathbb{N}}$ does not tend to zero.*

This observation has been exploited and combined with other techniques by many of their successors (see Chapter 7 for more details). In particular, in [109], def. 4.4 Banach algebras sharing the above property with $L(E)$ were called *retaining eigenvalues* because of the consequences for spectral theory.

However, it has turned out that not only spectral properties of elementary operators are closely related to hypothesis (*). In determining the qualities of an elementary operator S

sufficient conditions are usually derived from conditions on a_j and b_j by some commutativity assumptions. In order to infer from a property of S properties of the multiplying elements a_j and b_j, one needs some kind of non-commutativity. A natural obstacle which can occur is that the multiplication $M_{a,b}$ is zero without a or b being zero, and then (almost) nothing can be said about the properties of a or b. The hypothesis (*) excludes this possibility, but more - it gives an estimation downwards on the norm of $M_{a,b}$. In addition, a Banach algebra A satisfying (*) has to be very non-commutative (i.e. its center $Z(A)$ has to be trivial). All this made us suspect that one could use (*) as a basis on which a theory of elementary operators can be built.

The last four chapters of this thesis will endorse this supposition. Before describing their contents we enlarge upon the first three chapters and also give examples of applications of elementary operators.

In Chapter 1 we introduce the class of complex normed algebras which satisfy (*). We will call them *ultraprime algebras* because condition (*) is equivalent to the claim that an ultrapower of the normed algebra A is prime. Prime algebras are those in which non-zero elements a, b give rise to a non-zero multiplication $M_{a,b}$. This can be equivalently reformulated in terms of ideals of A, and hence one can apply ideal structure theory to study elementary operators. Some examples of ultraprime algebras as well as some basic properties, in particular their behaviour under various constructions giving new normed algebras, are discussed in the further development of Chapter 1. All prerequisites on ultrapowers of normed algebras that are needed have been collected in Appendix 1 to spare the reader

who is familiar with the theory of ultraproducts an unnecessary repetition. We would like to remark at this point that we will not use the language of Nonstandard Analysis nor its methods, and no proof depends on the ultrapower construction. But as a conceptual means, the notion of ultraprime algebras seems to facilitate the understanding of the underlying ideas and thus we have choosen this frame.

Chapter 2 contains the construction of a normed algebra of quotients of an ultraprime algebra which is again ultraprime. It takes some effort to reach this aim (Thm 2.15), but on the way the structure of ultraprime algebras is clarified. In particular, there are close relations to the continuity of module homomorphisms (Lemmas 2.3 and 2.9). Our effort will be rewarded with a fundamental result on elementary operators in Chapter 4 (Thm 4.2).

Special features of prime C*-algebras are discussed in the following chapter. It starts with a comparison of primeness and other conditions on the ideal structure, in particular primitivity of a C*-algebra. Proposition 3.1 contains a characterization of prime C*-algebras in terms of properties of multiplications and entails the fundamental result that every prime C*-algebra is ultraprime (Cor. 3.2). We also obtain the triviality of the extended centroid of a prime C*-algebra (Prop. 3.7) without the full construction carried out in Chapter 2. Thus the theory of elementary operators on prime C*-algebras can be developed without mentioning the concept of ultraprimeness (which is done in [111]). In fact, many of our results in Chapters 4 through 7 need the additional structure of a C*-algebra. Nevertheless, it seems appropriate to work in the more general frame of ultraprime Banach algebras in order to make it

clear how far reaching the basic idea is. Chapter 3 concludes with a new proof of an order-theoretical description of prime C*-algebras due to Archbold ([8], thm 3.7).

Elementary operators do not appear until Chapter 4. However, already in the first three chapters where we study the structure of ultraprime algebras it should become apparent in which way properties of multiplications have an impact on this structure. Generally, we take the point of view that properties of an elementary operator S on an algebra A reflect some of the properties of A (see e.g. Lemma 3.8 or 6.14.2), and it is therefore worthwhile to study elementary operators even if one is only interested in the structure of algebras.

In addition, elementary operators serve as small units from which more complicated operators can be built. For example, every normal completely positive contraction on the algebra $L(H)$ can be approximated by elementary operators of the form $\sum_j M_{c_j^*,c_j}$ ([96], thm 3.3), and every completely positive linear mapping T from a finite dimensional subfactor of a von Neumann algebra A into A is of the form $T = \sum_{j=1}^{n} M_{c_j^*,c_j}$ for some c_j in A ([68], prop. 2.1). Similarly every normal completely bounded A'-module homomorphism $T:L(H) \to L(H)$, where A is a von Neumann algebra on a Hilbert space H, can be approximated by elementary operators on A (an unpublished result by Haagerup).

On the other hand there are several results which represent certain classes of operators as elementary operators. For example, on every finite dimensional prime algebra each linear mapping is an elementary operator (cf. Cor. 4.4). Imposing some additional assumptions one obtains elementary operators of special types, e.g. if $T \in L(M_n)$ preserves the eigenvalues of

every $n \times n$-matrix then $T = M_{u,u^*}$ or $T = M_{u,u^*} \circ \tau$ where u in M_n is a unitary matrix and τ is the transposition on $n \times n$-matrices ([106], thm 3; see also [107]). This has been extended recently to surjective spectrum preserving linear maps on $L(E)$ by Jafarian and Sourour [89]. One may also recall Eidelheit's Theorem ([118], 2.5.19) stating that every automorphism of $L(E)$ is inner, and Sakai's Theorem ([121], 4.1.6) on the innerness of derivations on W*-algebras. An extension of the latter, viz. that every hermitian operator on a W*-algebra is a generalized inner derivation $\delta_{a,b}$, will be discussed in Chapter 4 (Prop. 4.12).

A third kind of applications of elementary operators is to other types of operators and related problems. Here, we mention a proof of Schauder's Theorem by compactness properties of $M_{a,b}$ ([20], 33.4). Others are discussed in [48], [120], and in Herrero's book [80]. Elementary operators also appear in the theory of operator convex functions and in Automatic Continuity Theory.

We now describe the results on elementary operators on ultraprime Banach algebras obtained in the second part of this thesis.

Chapter 4 is devoted to an investigation of the identity $S = 0$ and the closely connected question of the uniqueness of representation of an elementary operator by a sum of multiplications. This question will be answered completely on ultraprime normed algebras (Thm 4.2). In the proof of this fundamental theorem, the construction in Chapter 2 is used and the significance of ultraprimeness will become evident. Two alternative proofs for prime C*-algebras are then indicated. This is followed by the definition of the important notion of *minimal*

length of an elementary operator (Def. 4.6) and a characterization of minimal representations (4.7 and 4.8). It also follows from Theorem 4.2 that, for an ultraprime algebra A, the algebra $EL(A)$ of all elementary operators on A and the tensor product $A \otimes {}^{\circ}A$ of A with its reversed algebra ${}^{\circ}A$ are canonically isomorphic (Cor. 4.5). In the remainder of Chapter 4 we will apply Theorem 4.2 to several problems which reduce to an identity on elementary operators. In particular, we study hermitian operators on C*-algebras. In a supplement to Chapter 4 a result on derivations on prime rings is proved which has been deduced from their version of Theorem 4.2 for $L(E)$ by Fong and Sourour in [64] (this is however independent of 4.2).

In Chapter 5 the results of the previous chapter will be applied to obtain characterizations of completely positive elementary operators and of the extreme points of convex sets of completely positive elementary operators on prime C*-algebras (Thm 5.5 and Cor. 5.6). From this we also derive Theorem 5.7 which states that an elementary operator of minimal length n on a C*-algebra which is n-positive is already completely positive. This generalizes the result for the case $n = 1$ which was proved in [110].

As another application of Theorem 4.2 we give characterizations of compact and weakly compact elementary operators on prime C*-algebras in Chapter 6 (Thms 6.9 and 6.8). Special cases of this had been obtained previously by Ylinen [147], [149] and Akemann and Wright [2]. We also prove that on any C*-algebra without non-zero compact elements there are no non-zero weakly compact elementary operators (Cor. 6.12). This gives in particular an affirmative answer to a conjecture of Fong and Sourour [64] that on the Calkin algebra there are no

non-zero compact elementary operators. In a supplement to Chapter 6 we discuss several additional properties of the ideal $K(A)$ of compact elements in a C*-algebra A (6.14). For example we show that, if $K(A) \neq 0$, it is the only minimal closed ideal of a prime C*-algebra.

We begin the discussion of the spectra of elementary operators in Chapter 7 with a very simple result on invertibility of multiplications on unital rings (7.1). This is included mainly in order to illustrate which parts of the spectral theory will be easy and which ones deserve a more careful treatment. The results in 7.2 through 7.4, in 7.16 through 7.19 together with Proposition 3.1 yield a complete spectral analysis of multiplications on prime C*-algebras. Inclusions for the point spectra of an elementary operator S on a prime algebra and for the approximate point spectra of S on an ultraprime Banach algebra are derived in 7.5 through 7.7. The main result in Chapter 7 is Theorem 7.8 in which we obtain a description of the spectra of S on a prime unital C*-algebra. From this the classical result by Lumer and Rosenblum ([102], thm 10) is an immediate consequence (Cor. 7.11). Two other important results are Theorem 7.13, where we describe the joint spectrum of left and right multiplications on prime C*-algebras which was recently discussed again for $L(H)$ in [40], and Theorem 7.20 which gives criteria for ultraweak denseness of the range of $\rho - S$ on a prime W*-algebra.

In order to avoid a long list of definitions and basic properties all necessary prerequisites on joint spectra are collected in Appendix 2. Here we also recall relations betweer the spectra of n-tuples in a unital Banach algebra A and in some ultrapower $\hat{A}_U$ of A.

This thesis is not intended to give a comprehensive account of the theory of elementary operators. Several important questions are not tackled, e.g. Fredholm properties and the index of elementary operators (for this see the concluding remarks in Chapter 7), or the problem of determining the norm of an elementary operator (for a description of the norm of a multiplication on a C*-algebra see [109], cor. 4.9). There are also recent contributions dealing with the spectrality and the decomposability of elementary operators ([85], [86], [3]) which will not be discussed. We simply tried to provide a unified approach to the theory of elementary operators, and to this end one should first study the prime case. Extensions of our results to arbitrary (C*-)algebras mainly depend on a suitable decomposition theory, and are sometimes obvious (like in Theorem 5.7), sometimes harder to achieve. An example of an extension using reduction theory of W*-algebras is given in [109].

We conclude this introduction with some notation which will be used without further explanation in the sequel. Most of the symbols occuring in the text are listed in the Index of Symbols with the page(s) of their definition (some of them are met with different meanings).

Notation. If E, F are normed spaces, $L(E,F)$ denotes the space of all bounded operators from E into F and $L(E)$ is the normed algebra of all bounded operators on E. Usually H stands for a Hilbert space (not necessarily separable), thus $L(H)$ is the W*-algebra of all bounded operators on H. If ξ,η are non-zero vectors in H then $\xi \otimes \eta$ denotes the corresponding rank-one operator $\zeta \mapsto (\zeta|\eta)\xi$. The linear hull of a subset X of a vector space is written as $\operatorname{lin} X$. All vector spaces and algebras are defined over the field $\mathbb{C}$.

1. THE CONCEPT OF ULTRAPRIME BANACH ALGEBRAS

In this chapter we lay the foundation on which our work is built. We introduce the notion of an ultraprime normed algebra and discuss some of its basic properties as well as its behaviour under various standard algebraic constructions.

Recall first the notion of a prime ring. Let R be an arbitrary ring, and let I be an ideal in R (if not specified, by an ideal we always mean a two-sided ideal). Then I is said to be a *prime ideal* if, whenever I_1, I_2 are ideals in R with $I_1 I_2 \subseteq I$, we have $I_1 \subseteq I$ or $I_2 \subseteq I$. If the zero ideal 0 is a prime ideal, then R is called a *prime ring*. Consequently, I is a prime ideal if and only if R/I is a prime ring.

A commutative ring R is, of course, prime if and only if R is an integral domain. The ring M_n of all complex $n \times n$-matrices provides an example of a non-commutative prime ring which contains non-zero divisors of zero. The most important class of prime rings is that of *primitive* rings ([88], prop. VIII.2.1), i.e. those which have a faithful irreducible left module.[1] In general, a prime ring need not be primitive; $\mathbb{Z}$, the ring of integers, is the simplest example. There are several conditions which ensure that primeness implies primitivity, for instance the existence of minimal idempotents. (Others will be given in subsequent chapters.)

Although general prime rings had appeared occasionally in the literature before 1949, it seems to have been N.H. McCoy

---

[1] Thus, *primitivity* will always mean *left primitivity*.

who initiated a systematic investigation of these objects by his paper [112]. Since then prime rings have played an important rôle, mainly in Algebra. Closed prime ideals were also studied in topological rings (see e.g. [150]) and have found applications in various branches of Analysis. For example, they proved to be useful in Automatic Continuity Theory ([14], [43]); see also Lemma 2.3). Nonetheless, they were mostly replaced by primitive ideals which can be treated more easily and behave better under topological manipulations. Our objective here is to define a class of prime ideals well compatible with topology.

To this end we note at first that in the definition of prime ideal one can replace the "ideals" I_1, I_2 by either (i) left ideals, (ii) right ideals, (iii) principal ideals, or (iv) I_1 a right ideal and I_2 a left ideal, and obtains the same definition ([112], thm 1). If R is a topological ring, one can also use *closed* (left, right, principal) ideals instead and obtains again an equivalent definition. Therefore, this gives no possibility for a kind of 'topological primeness'. However, as already observed by McCoy([112], thm 1), there is still another equivalent formulation. This one will play the major rôle in our approach, since firstly it can be tightened to a topological version, and secondly it reveals the connection between prime algebras and elementary operators.

For any two elements a and b in a ring R let $M_{a,b}$ denote the additive map $x \to axb$ on R. We call $M_{a,b}$ a (*two-sided*) *multiplication on* R. We have

The ring R *is prime if and only if* $M_{a,b}$ *is non-zero whenever both* a *and* b *are non-zero.*

We give the simple and well known proof of this first fundamental step in our approach to elementary operators.

Suppose that R is prime and take $a, b \in R \setminus \{0\}$. If $M_{a,b} = 0$, then $RaRRbR = 0$ so that $RaR = 0$ or $RbR = 0$. In the first case, let (a) denote the principal ideal generated by a. Since $(a)^3 \subseteq RaR = 0$ and R is prime, it follows that $(a) = 0$, thus $a = 0$. The second case is treated similarly. Therefore, $M_{a,b} = 0$ implies $a = 0$ or $b = 0$. Suppose now that R is a ring which satisfies this latter condition for all $a, b \in R$. If I_1 and I_2 are non-zero ideals in R, take $a \in I_1 \setminus \{0\}$ and $b \in I_2 \setminus \{0\}$. Then there exists $x \in R$ such that $0 \neq axb \in I_1 I_2$. Thus, R is a prime ring.

So far we worked in a purely algebraic setting. From now on we will exclusively be concerned with *normed algebras* (with exception of Supplement 4.17 and Proposition 7.1) because we are mainly interested in *bounded linear* mappings.

For the remainder of this chapter A will denote a complex normed algebra. As just observed, A is prime if and only if the bilinear mapping

$$M: A \times A \to L(A) , \quad (a,b) \mapsto M_{a,b}$$

is injective in each variable separately, i.e. $(a,.) \mapsto M_{a,.}$ and $(.,b) \mapsto M_{.,b}$ are injective for each $a, b \in A \setminus \{0\}$. Our topological version of primeness now is defined by demanding that M is 'separately bounded below'.

We first state a lemma which relates this to an ultrapower $\hat{A}_U$ of A. The reader is requested to familiarize him/herself with this notion (see Appendix 1).

Lemma 1.1. *Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. The following conditions are equivalent:*

(a) *For any pair $((x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}})$ of sequences in A such that $\|x_k\| = \|y_k\| = 1$ for all $k \in \mathbb{N}$ there exists a bounded sequence $(z_k)_{k\in\mathbb{N}}$ in A such that $(x_k z_k y_k)_{k\in\mathbb{N}}$ does not tend to zero.*

(b) *There exists a constant $c_M > 0$ such that for all $a,b \in A$ $\quad \|M_{a,b}\| \geq c_M \|a\| \|b\|$.*

(c) *The algebra $\hat{A}_{\mathcal{U}}$ is prime.*

Proof. (a) => (b) If there is no constant $c_M > 0$ with the desired property, then, for each $k \in \mathbb{N}$, there are elements $a_k, b_k \in A \setminus \{0\}$ such that $\|M_{a_k,b_k}\| < \frac{1}{k} \|a_k\| \|b_k\|$. By putting $x_k = a_k/\|a_k\|$, $y_k = b_k/\|b_k\|$ we obtain sequences of unit vectors such that $\lim_{k\to\infty} \|M_{x_k,y_k}\| = 0$. Thus $x_k z_k y_k \to 0$ for every bounded sequence $(z_k)_{k\in\mathbb{N}}$.

(b) => (c) Let $\hat{a}, \hat{b} \in \hat{A}_{\mathcal{U}}$ with $\|\hat{a}\| = \|\hat{b}\| = 1$. By A.1.2, there are representing sequences $(a_k)_{k\in\mathbb{N}} \in \hat{a}$ and $(b_k)_{k\in\mathbb{N}} \in \hat{b}$ such that $\|a_k\| = \|b_k\| = 1$ for all $k \in \mathbb{N}$. Since, by hypothesis, $\|M_{a_k,b_k}\| \geq c_M$ for all k, there is a sequence $(z_k)_{k\in\mathbb{N}}$ with $\|z_k\| = 1$ and $\|a_k z_k b_k\| \geq c_M - \frac{1}{k}$ for all $k \in \mathbb{N}$. With $\hat{z} = (z_k)_{\mathcal{U}}$ we obtain $\|\hat{a}\hat{z}\hat{b}\| = \lim_{\mathcal{U}} \|a_k z_k b_k\| \geq c_M$, hence $\hat{A}_{\mathcal{U}}$ is prime.

(c) => (a) If $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ are sequences of unit vectors in A put $\hat{x} = (x_k)_{\mathcal{U}}$, $\hat{y} = (y_k)_{\mathcal{U}} \in \hat{A}_{\mathcal{U}} \setminus \{0\}$. Any sequence $(z_k)_{k\in\mathbb{N}} \in \hat{z}$, where $\hat{z} \in \hat{A}_{\mathcal{U}}$ is choosen such that $\hat{x}\hat{z}\hat{y} \neq 0$, has the desired property. □

Definition 1.2. Let I be a closed ideal of the normed algebra A. Then $\hat{I}_U$ is a closed ideal of the Banach algebra $\hat{A}_U$ by A.1.9, whenever U is a free ultrafilter on $\mathbb{N}$. We call I an *ultraprime ideal* if $\hat{I}_U$ is a prime ideal in $\hat{A}_U$. Moreover A is called an *ultraprime algebra* if $\hat{A}_U$ is a prime algebra.

By Lemma 1.1 and the following result, this definition is consistent and does not depend on the choice of the ultrafilter U.

Lemma 1.3. *The closed ideal I is an ultraprime ideal if and only if $A_{/I}$ is an ultraprime algebra.*

Proof. By A.1.10, $(A_{/I})\hat{}_U = \hat{A}_{U/\hat{I}_U}$. Thus $A_{/I}$ is an ultraprime algebra if and only if $\hat{I}_U$ is a prime ideal.□

Corollary 1.4. *Every ultraprime ideal is a closed prime ideal.*

Proof. If I is an ultraprime ideal, then $A_{/I}$ is an ultraprime and therefore a prime algebra by 1.3 and 1.1.□

This corollary ensures that ultraprimeness is in fact a stronger property than primeness. Its statement can be reversed for C*-algebras (3.2).

Remarks 1.5. 1. One could omit the demand *closed* ideal in Definition 1.2 and would obtain the same notion for not necessarily closed ideals (see the Remark in A.1.10 and Proposition 1.6). However we feel it more natural to work with closed ideals.

2. It appears noteworthy that, although $\hat{T} \in L(\hat{E}_F, \hat{F}_F)$ is bounded below whenever $T \in L(E,F)$ is bounded below and $\hat{E}_F$, $\hat{F}_F$ are the reduced powers of the normed spaces E, F respec-

tively with respect to any filter F finer than the Fréchet filter on $\mathbb{N}$ (compare A.2.8), this does not remain true for bilinear mappings unless $F = U$ is a free ultrafilter on $\mathbb{N}$. For an example take $A = \mathbb{C}$ (which is clearly an ultraprime algebra) and let c_o denote the space of complex null sequences. Then $\ell^\infty(A)/c_o$ contains non-zero divisors of zero which means that $\hat{M}$ is even not separately injective.

3. In [109], Def. 4.4 a Banach algebra A satisfying condition (a) in 1.1 was called 'retaining eigenvalues' because of its consequences for spectral theory. This condition has been used explicitly for the first time by Lumer and Rosenblum in [102], thm 9. However, our new description as ultraprime algebras clarifies the intimate relation with the ideal structure of A.

We are now going to give some examples of ultraprime normed algebras. Note that an ultraprime algebra A has trivial center $Z(A)$. In fact, if $\dim Z(A) \geq 2$, then there is a topological divisor of zero x in $Z(A)$ with $\|x\| = 1$ ([152], 14.4). If $(y_k)_{k\in\mathbb{N}}$ is a sequence of unit vectors in A with $\lim_{k\to\infty} xy_k = 0$ there cannot exist a bounded sequence $(z_k)_{k\in\mathbb{N}}$ with $xz_ky_k \neq 0$. Therefore, $\{0\}$ and $\mathbb{C}$ are the only commutative complex ultraprime normed algebras. From this we also deduce the existence of prime normed algebras that are not ultraprime, for example the disc algebra $A(\Delta)$ and the convolution algebra $L^1(\mathbb{R}_+)$. (Observe, however, that $L^1(G)$ is not prime for every non-trivial locally compact group G [151].) Thus, ultraprimeness is a strictly stronger notion than primeness. One may modify these examples to obtain algebras which are also radical ([20], ex.18.25; [47]), or non-commutative and

prime, but not ultraprime [20], ex. 47.8). Another example of a non-commutative prime algebra, which is not ultraprime, is a non-commutative version of the disc algebra: $A(U,B)$, the Banach algebra of all functions f which are holomorphic in the interior of the unit ball U in a complex Banach space E, continuous on U, and have values in a non-commutative complex Banach algebra B with non-trivial center $Z(B)$. Since $A(U,B)$ has no non-zero divisors of zero, it is prime, and since all functions with values in $Z(B)$ lie in $Z(A(U,B))$, it is not ultraprime.

If A is a finite dimensional prime algebra, then A is ultraprime since $A = \hat{A}_U$. The algebra $L(E)$ of all bounded linear mappings on a normed space E is ultraprime since $\|M_{a,b}\| = \|a\| \|b\|$ for all $a,b \in L(E)$. More generally, every subalgebra A of $L(E)$ which contains the ideal $F(E)$ of finite rank operators enjoys the same property and thus is ultraprime. We will prove in Chapter 3 that a C*-algebra which is prime is already ultraprime.

Similar to primitive and prime algebras, ultraprime algebras behave well under several constructions giving new normed algebras. We now discuss some of the basic methods and postpone the more elaborate construction of an algebra of quotients to Chapter 2. We start with two topological procedures to enlarge a normed algebra. The first of them shows in particular that the completion of an ultraprime normed algebra is an ultraprime Banach algebra.

Proposition 1.6. *Let* A *be a subalgebra of the normed algebra* B. *Then* A *is ultraprime if and only if its closure* $\bar{A}$ *is ultraprime.*

Proof. This is immediate from the fact that $\hat{A}_U = \hat{\hat{A}}_U$ which is a consequence of A.1.6.□

Observe that Proposition 1.6 fails for prime normed algebras: The algebra of all polynomials on the unit interval contains no non-zero divisors of zero, while its closure in the supremum norm, the algebra of all continuous functions, is not prime. The next result shows that the class of ultraprime algebras is also stable under ultrapowers.

Proposition 1.7. *Let* A *be an ultraprime normed algebra. Then every ultrapower* $\hat{A}_U$ *of* A *is again ultraprime.*

Proof. If U and V are free ultrafilters on $\mathbb{N}$ there is another free ultrafilter W on $\mathbb{N}$ such that $(\hat{A}_U)\hat{}_V = \hat{A}_W$ (A.1.5). Thus the assertion follows. Alternatively, one may analyze the proof of Lemma 1.1 to see that condition (b) is inherited by any ultrapower $\hat{A}_U$.□

There are few possibilities of making an ultraprime algebra smaller without loss of ultraprimeness. We note the following.

Proposition 1.8. *Let* I *be an ideal in the ultraprime normed algebra* A. *Then* I *is an ultraprime algebra.*

Proof. By A.1.9, $\hat{I}_U$ is an ideal in the prime algebra $\hat{A}_U$. Therefore, the result follows from the corresponding result for ideals in prime rings ([112], lem. 2).□

We add two more standard constructions of enlarging an algebra.

Proposition 1.9. *A normed algebra* A *without identity is ultraprime if and only if its unitization* $A + \mathbb{C}$ *is ultraprime.*

Proof. Since A is an ideal in $A + \mathbb{C}$, the "if"-part follows from the preceding proposition. By A.1.11, $(A + \mathbb{C})\hat{}_U = \hat{A}_U + \mathbb{C}$ is the unitization of $\hat{A}_U$. Thus, the "only if"-part is a consequence of the fact that the unitization of a prime algebra is still prime (e.g. [8], lem. 3.4).□

Proposition 1.10. *Let* A *be a unital ultraprime normed algebra. Then the* $n\times n$*-matrix algebra* $M_n(A)$ *is also ultraprime.*

Proof. Take a free ultrafilter U on $\mathbb{N}$. By A.1.12, we have $M_n(A)\hat{}_U = M_n(\hat{A}_U)$. Since $M_n(\hat{A}_U)$ is prime, by Definition 1.2 and [112], thm 8, it follows that $M_n(A)$ is ultraprime.□

It is easy to see that the converse of Proposition 1.10 also holds.

An important, but still unsolved problem is to describe precisely the relations between primitivity and ultraprimeness of a Banach algebra A. We merely remark that the set-theoretic ultrapower of a primitive ring is again a primitive ring ([81], thm 5.5). We do not know an example of a primitive Banach algebra A with an ultrapower $\hat{A}_U$ which is not primitive.

2. THE BANACH ALGEBRA OF QUOTIENTS OF AN ULTRAPRIME BANACH ALGEBRA

A special feature of a commutative integral domain is the existence of a ring of quotients which is a field. In the setting of commutative Banach algebras, Suciu studied in [130], [131] algebras of quotients which can be normed. For non-commutative rings there are several constructions of rings of quotients, the most important of them is the *maximal* or *Utumi's* ring of quotients ([82], [99], [142]). It has found many applications, for instance in the theory of prime rings satisfying a generalized polynomial identity. Starting with an arbitrary prime Banach algebra A over $\mathbb{C}$, it is however not always possible to endow Utumi's ring of quotients with a norm. Since the center of this ring of quotients is a field it would have to be the complex numbers by Mazur's theorem. Therefore, at least the triviality of the center of A has to be assumed in addition.

In this chapter we present a construction of an algebra of quotients for any ultraprime normed algebra, which will again be an ultraprime normed algebra. This result will be applied to elementary operators in Chapter 4.

Throughout this chapter A will denote a non-zero ultraprime normed algebra over $\mathbb{C}$ if not otherwise specified. Informally, its 'algebra of quotients' Q_A will consist of equivalence classes of continuous right A-module homomorphisms from ideals of A into A. We therefore prepare the construction of Q_A by some remarks on ideals and module homomorphisms.

2.1. Let $\mathcal{I}$ respectively $\mathcal{I}_c$ denote the set of all non-zero ideals respectively all non-zero closed ideals of A. Both $(\mathcal{I},\subseteq)$ and $(\mathcal{I}_c,\subseteq)$ are order complete lattices since A is prime. If A is a C*-algebra then, for $I_1, I_2 \in \mathcal{I}_c$, $I_1 \wedge I_2 = I_1 I_2$ and $I_1 \vee I_2 = I_1 + I_2$ ([35], 5.1 and 6.2). By considering the elements of $\mathcal{I}$ as normed right A-submodules of the right A-module A we define

$$\mathcal{M} = \{(I,f) \mid I \in \mathcal{I},\ f:I \to A \text{ is a continuous right } A\text{-module homomorphism}\}$$

and

$$\mathcal{M}_c = \{(I,f) \mid I \in \mathcal{I}_c,\ f:I \to A \text{ is a continuous right } A\text{-module homomorphism}\}.$$

In general, $\mathcal{M}_c \subset \mathcal{M}$, and we now add some results on the continuity of A-module homomorphisms.

2.2. First of all we note that every right A-module homomorphism $f:I \to A$ is linear. For

$$(f(\lambda x) - \lambda f(x))a = f(\lambda xa) - f(x\lambda a) = 0$$

for all $x \in I$, $a \in A$ and $\lambda \in \mathbb{C}$. The primeness of A now proves the assertion.

Next we observe that for an arbitrary C*-algebra A every right A-module homomorphism f from a closed ideal I of A into A is automatically continuous, and hence corresponds uniquely to a left multiplier of I. This can be proved by standard methods (see [116], 3.12.2 and 3.12.3). More generally, right A-module homomorphisms on a Banach algebra A are left centralizers of A studied by B.E.Johnson [90]. A slight modification of his argument (ibid., thm 14) shows that every

A-bimodule homomorphism from a *closed* ideal of a *prime Banach* algebra into itself is automatically continuous, too.

In addition we have the following result which will be crucial in the sequel.

Lemma 2.3. *Every non-zero* A*-bimodule homomorphism* $f:I \to A$ *from an ideal* I *of an ultraprime normed algebra* A *into* A *is a topological isomorphism onto its image.*

Proof. For all $x, y \in I$ we have

$$\|x\| \, \|f(y)\| \geq \|M_{x,f(y)}\| = \|M_{f(x),y}\| \geq c_M \, \|f(x)\| \, \|y\|$$

and

$$\|f(x)\| \, \|y\| \geq \|M_{f(x),y}\| = \|M_{x,f(y)}\| \geq c_M \, \|x\| \, \|f(y)\|$$

for some $c_M > 0$ by Lemma 1.1.

By the first inequality, $f \in L(I,A)$ and thus, with $\|f\| = \sup_{\|y\|=1} \|f(y)\|$, we obtain

$$c_M \, \|f\| \, \|x\| \leq \|f(x)\| \leq \frac{\|f\|}{c_M} \, \|x\|$$

for all $x \in I$. This completes the proof.□

The surprisingly strong conclusion of Lemma 2.3 will find its explanation by Corollary 2.13.

Our next objective is to define an equivalence relation on $\mathfrak{M}$.

Proposition 2.4. *The relation* $(I,f) \sim (J,g)$ *if there is* U *in* I *such that* $U \subseteq I \cap J$ *and* $f = g$ *on* U *is an equivalence relation on* $\mathfrak{M}$.

Proof. It is obvious that $\sim$ is both reflexive and symmetric. If $(I_1,f_1) \sim (I_2,f_2) \sim (I_3,f_3)$ and $U, V \in \mathcal{I}$ are such that $U \subseteq I_1 \cap I_2$, $V \subseteq I_2 \cap I_3$, $f_{1|U} = f_{2|U}$ and $f_{2|V} = f_{3|V}$ then $U \cap V \in \mathcal{I}$, since A is prime, $U \cap V \subseteq I_1 \cap I_3$ and $f_{1|U\cap V} = f_{3|U\cap V}$. Thus $(I_1,f_1) \sim (I_3,f_3)$, and $\sim$ is also transitive.□

Next we prove that in the definition of $\sim$, the ideal U can be taken to be $I \cap J$.

Lemma 2.5. *If* $(I,f) \sim (J,g)$ *then* $f_{|I\cap J} = g_{|I\cap J}$.

Proof. Put $h = f - g$ and take $U \in I$ with $h(U) = 0$. For $x \in I \cap J$, $y \in U$ we have $h(x)y = h(xy) = 0$, i.e. $h(I \cap J)U = 0$. Since $h(I \cap J)$ is a right ideal of A and A is prime, it follows that $h(I \cap J) = 0$. Thus $f_{|I\cap J} = g_{|I\cap J}$. □

If $f:I \to A$ is an A-bimodule homomorphism and f is not injective, then $U = \ker f \in \mathcal{I}$. Thus $(I,f) \sim (I,0)$, and the preceding lemma shows that $f = 0$. Therefore, a non-zero A-bimodule homomorphism from an ideal of a *prime* algebra A into A has to be injective (compare with Lemma 2.3 where A is supposed to be *ultraprime*).

Let $Q(A) = \mathcal{M}/_{\sim}$ be the set of all equivalence classes and denote the equivalence class of $(I,f) \in \mathcal{M}$ by $[(I,f)]$.

Proposition 2.6. *If we define the following addition and multiplications on* $Q(A)$

$$[(I,f)] + [(J,g)] := [(I \cap J, f + g)],$$

$$\lambda\,[(I,f)] := [(I,\lambda f)],$$

$$[(I,f)] \cdot [(J,g)] := [(JI,fg)].$$

for $(I,f), (J,g) \in \mathcal{M}$ *and* $\lambda \in \mathbb{C}$,
then $Q(A)$ *becomes a prime unital algebra over* $\mathbb{C}$ *containing an isomorphic image of* A. *The center* $Z(Q(A))$ *is a field.*

Proof. As an illustration of the techniques involved we prove that the algebra multiplication is well-defined. Let (I,f),

$(J,g) \in \mathcal{M}$. Then $(JI,fg) \in \mathcal{M}$ since A is prime. If $(I_1,f_1) \sim (I,f)$ and $(J_1,g_1) \sim (J,g)$ then $f_{1|I_1 \cap I} = f_{|I_1 \cap I}$ and $g_{1|J_1 \cap J} = g_{|J_1 \cap J}$ by Lemma 2.5. Since $U = (J_1 \cap J)(I_1 \cap I) \subseteq J_1 I_1 \cap JI$ and $f_1 g_{1|U} = fg_{|U}$ it follows that $(J_1 I_1, f_1 g_1) \sim (JI,fg)$, and thus the multiplication is independent of the choice of the representatives.

The axioms of a complex algebra are easily verified. The identity in $Q(A)$ is the equivalence class $[(A,id)]$ of the identical homomorphism on A.

Each element $a \in A$ can be identified with the equivalence class $[(A,L_a)] \in Q(A)$, where $L_a x := ax$. For, if $[(A,L_a)] = 0$ then there is $U \in I$ such that $aU = 0$. Hence, $axb = 0$ whenever $x \in A$, $b \in U \setminus \{0\}$, and the primeness of A now yields $a = 0$. From now on we consider A as a subalgebra of $Q(A)$.

Let $q_1 = [(I_1,f_1)]$, $q_2 = [(I_2,f_2)] \in Q(A) \setminus \{0\}$. Then there are $x_1 \in I_1$, $x_2 \in I_2$ such that $q_1 x_1 = f_1(x_1) \in A \setminus \{0\}$ and $q_2 x_2 = f_2(x_2) \in A \setminus \{0\}$. Since A is prime there is $z \in A$ for which $q_1 x_1 z q_2 x_2 \neq 0$, hence $q_1 x_1 z q_2 \neq 0$. Therefore, $M_{q_1,q_2} \neq 0$ whenever $q_1,q_2 \in Q(A) \setminus \{0\}$ which proves the primeness of $Q(A)$.

Finally, we show that $Z(Q(A))$ is a field. Observe that $Z(Q(A)) = \{[(I,f)] \in Q(A) \mid f \text{ is an } A\text{-bimodule homomorphism}\}$. In fact, if $f:I \to A$ is an A-bimodule homomorphism on $I \in \mathcal{I}$ and $(J,g) \in \mathcal{M}$ then, for all $x \in I$, $y \in J$,

$$fg(yx) = f(g(yx)) = f(g(y)x) = g(y)f(x) = g(yf(x)) = g(f(yx)) = gf(yx),$$

i.e. $fg = gf$ on JI. Hence $[(I,f)] \in Z(Q(A))$. Conversely, if $(I,f) \in q \in Z(Q(A))$, for every $a \in A$ $[(I,f)] \cdot [(A,L_a)] = [(A,L_a)] \cdot [(I,f)]$. Hence $f \circ L_a = L_a \circ f$ on I proving that

f is a left A-module homomorphism, too.

Take $q = [(I,f)] \in Z(Q(A)) \setminus \{0\}$. By Lemma 2.3, $f(A) \in \mathcal{I}$ and $f^{-1}: f(A) \to A$ is a continuous A-bimodule homomorphism. Thus $q^{-1} = [(f(A), f^{-1})]$. □

Remark 2.7. So far we did not make essential use of our hypotheses that A is ultraprime and all module homomorphisms are continuous. In fact, our construction of $Q(A)$ is closely related to Utumi's ring of quotients: this one is obtained by extending the relation $\sim$ to $\mathcal{M}_a$, the set of all pairs (I,f) where $I \in \mathcal{I}$ and $f: I \to A$ is an arbitrary right A-module homomorphism, and defining $Q_a(A) = \mathcal{M}_{a/\sim}$ (see [82], ch. 1.3). We could also restrict $\sim$ to $\mathcal{M}_c$ and define $Q_c(A) = \mathcal{M}_{c/\sim}$; then, Proposition 2.6 remains true with the following modification in the definition of the multiplication

$$[(I,f)] \cdot [(J,g)] := [(\overline{JI}, fg)].$$

However, we have to assume that A is an ultraprime Banach algebra in order to be able to extend fg from JI to $\overline{JI}$. Clearly, one has $Q_c(A) \subseteq Q(A) \subseteq Q_a(A)$, but since $(\overline{I},f) \sim (I,f)$ for every $(I,f) \in \mathcal{M}$ we actually have $Q_c(A) = Q(A)$ for every ultraprime Banach algebra A. Because of automatic continuity (cf. 2.2), for a prime C*-algebra A we could also start with

$$\mathcal{M}^c = \{(I,f) \mid I \in \mathcal{I}_c,\ f: I \to A \text{ is a right } A\text{-module homomorphism}\}.$$

Clearly, $\mathcal{M}^c{}_{/\sim} = Q_c(A)$ in this case.

We call $Q(A)$ the *ring of quotients of the ultraprime algebra* A. In view of Lemma 2.3, we have $Z(Q(A)) = Z(Q_a(A))$, which is called the *extended centroid* of A ([82], p. 22).

Now the significance of continuous module homomorphisms and of the ultraprimeness of A will become apparent.

First we have a closer look at the elements of $\mathcal{Q}(A)$.

2.8. Let $q \in \mathcal{Q}(A)$. We define

$$(I_1,f_1) \leq (I_2,f_2) \quad \text{if} \quad I_1 \subseteq I_2,$$

whenever (I_1,f_1), $(I_2,f_2) \in q$. By Lemma 2.5, this implies that f_2 is an extension of f_1. The relation $\leq$ is clearly reflexive and transitive. If $(I_1,f_1) \leq (I_2,f_2) \leq (I_1,f_1)$ then $I_1 = I_2$ and $f_1 = f_2$, thus $\leq$ is also antisymmetric. Therefore the set $\{(I,f) \mid (I,f) \in q\}$ is partially ordered by $\leq$. We call $\leq$ the *canonical order on* q.

Our next aim is to define a (semi-)norm on $\mathcal{Q}(A)$. Before doing so, we need another result on the continuity of A-module homomorphisms.

Lemma 2.9. *Let* A *be an ultraprime normed algebra. If*

$$\mathfrak{M} \ni (I,f) \sim (J,g) \in \mathfrak{M}_a \quad \textit{then} \quad (J,g) \in \mathfrak{M}.$$

Proof. Suppose that $g:J \to A$ is not continuous. Then there is a null sequence $(x_k)_{k\in\mathbb{N}}$ in J such that $\|g(x_k)\| = 1$ for all k. Take some $y \in I \cap J$ with $\|y\| = 1$. By Lemma 1.1, there is a bounded sequence $(z_k)_{k\in\mathbb{N}}$ in A with the property that $(g(x_k)z_ky)_{k\in\mathbb{N}}$ does not tend to zero. On the other hand, $f_{|I \cap J} = g_{|I \cap J}$ by Lemma 2.5. Thus $g(x_k)z_ky = g(x_kz_ky) = f(x_kz_ky) \to 0$, a contradiction. Therefore $(J,g) \in \mathfrak{M}$. □

Corollary 2.10. *If* $q \in \mathcal{Q}_a(A)$ *has a representative* $(I,f) \in \mathfrak{M}$ *then* $q \in \mathcal{Q}(A)$.

Proposition 2.11. *Let* $q \in \mathcal{Q}(A)$ *where* A *is an ultraprime normed algebra. Under the canonical order* $(q,\leq)$ *is an order complete lattice containing a largest element* $\sup q$.

Proof. Let (I_1,f_1), $(I_2,f_2) \in q$. Then $I_1 \cap I_2 = I_1 \wedge I_2$ belongs to $\mathcal{I}$ (by 2.1) and $f_1|_{I_1 \cap I_2} = f_2|_{I_1 \cap I_2}$ (by 2.5). Thus $(I_1,f_1) \wedge (I_2,f_2) = (I_1 \cap I_2, f_1|_{I_1 \cap I_2})$ belongs to q.

If $q_o \subseteq q$ is an arbitrary non-void subset of q with lower bound $(I_o,f_o) \in q$, we put $I_{min} := \bigcap \{ I \mid (I,f) \in q_o \}$ and $f_{min} := f|_{I_{min}}$ where f is arbitrary with $(I,f) \in q_o$. Since $I_o \subseteq I_{min}$, (I_{min},f_{min}) is the infimum of q_o in q.

We now define the supremum of (I_1,f_1), $(I_2,f_2) \in q$. If $x \in I_1 \cup I_2$ we put

$$f_1 \vee f_2\,(x) := \begin{cases} f_1(x) & \text{if } x \in I_1 \\ f_2(x) & \text{if } x \in I_2 \end{cases} .$$

Then $f_1 \vee f_2$ is well-defined by Lemma 2.5.

Take now $x \in I_1 + I_2$. If x is represented by $x = x_1 + x_2 = x_1' + x_2'$ with $x_1, x_1' \in I_1$ and $x_2, x_2' \in I_2$ then $x_1 - x_1' = x_2' - x_2 \in I_1 \cap I_2$. By 2.5, $f_1(x_1 - x_1') = f_2(x_2' - x_2)$, i.e. $f_1(x_1) + f_2(x_2) = f_1(x_1') + f_2(x_2')$.

Thus we may define $f_1 \vee f_2 : I_1 + I_2 \to A$ by $f_1 \vee f_2\,(x) = f_1(x_1) + f_2(x_2)$ and obtain a right A-module homomorphism. It follows by Lemma 2.9 that $f_1 \vee f_2$ is continuous and so $(I_1,f_1) \vee (I_2,f_2) = (I_1 + I_2, f_1 \vee f_2)$.

By means of this we obtain the existence of suprema of finite subsets of q.

Let now p be a non-void subset of q (we do not assume that p is bounded above). We put $I_{(p)}$ the ideal generated by all ideals I such that $(I,f) \in p$. Then every element $x \in I_{(p)}$ has a representation $x = \sum_{i \in p_1} x_i$ where p_1 is a finite subset of p and $x_i \in I_i$ with $(I_i,f_i) \in p_1$. If $x = \sum_{j \in p_2} x_j'$ is another such representation of x and $(\bigvee_{i \in p_1} I_i, \bigvee_{i \in p_1} f_i)$ respec-

tively $(\bigvee_{j\in p_2} I_j, \bigvee_{j\in p_2} f_j)$ are the suprema of p_1 resp. p_2, we have

$$\sum_{i\in p_1} f_i(x_i) = \bigvee_{i\in p_1} f_i(\sum_{i\in p_1} x_i) = \bigvee_{j\in p_2} f_j(\sum_{j\in p_2} x'_j) = \sum_{j\in p_2} f_j(x'_j)$$

since $x \in \bigvee_{i\in p_1} I_i \cap \bigvee_{j\in p_2} I_j$ and $(\bigvee_{i\in p_1} I_i, \bigvee_{i\in p_1} f_i) \sim (\bigvee_{j\in p_2} I_j, \bigvee_{j\in p_2} f_j)$.

Thus by $f_{(p)}(x) := \sum_{i\in p_o} f_i(x_i)$ where $x = \sum_{i\in p_o} x_i$ is any representation of x with $x_i \in I_i$, $(I_i, f_i) \in p_o$ and $p_o \subseteq p$ finite, we obtain a well-defined right A-module homomorphism $f_{(p)}: I_{(p)} \to A$ which is continuous by Lemma 2.9. Clearly, $(I_{(p)}, f_{(p)})$ is the supremum of p.
In particular, there is a largest element $\sup q := (I_{(q)}, f_{(q)})$ in $(q, \leq)$.□

One might be tempted to think of q as being represented by $\sup q$ and to define $\|q\|$ as $\|f_{(q)}\|$. This is however not possible since in general $I_{(q_1)} \cap I_{(q_2)} \subsetneq I_{(q_1+q_2)}$ and also $I_{(q_2)} I_{(q_1)} \subsetneq I_{(q_1 q_2)}$. For example if q is invertible in $\mathcal{Q}(A)$ with $I_{(q)} \neq A$, then $I_{(q)} \cap I_{(-q)} = I_{(q)} \subsetneq I_{(q-q)} = A$ and $I_{(q^{-1})} I_{(q)} \subseteq I_{(q)} \subsetneq I_{(qq^{-1})} = A$. Therefore, the order is not fully compatible with the algebraic structure of $\mathcal{Q}(A)$.

We now continue to construct $\mathcal{Q}_A$.

Lemma 2.12. *By defining* $\|q\| = \inf\{\|f\| \mid (I,f) \in q\}$ *we obtain an algebra semi-norm on* $\mathcal{Q}(A)$. *Its restriction to either* $Z(\mathcal{Q}(A))$ *or* A *is a norm.*

Proof. Let $q, q_1, q_2 \in \mathcal{Q}(A)$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}\|\lambda q\| &= \inf\{\|g\| \mid (J,g) \in \lambda q\} \\ &\leq \inf\{\|\lambda f\| \mid (I,f) \in q\} = |\lambda| \, \|q\|.\end{aligned}$$

But, if $(J,g) \sim (I,\lambda f)$ then $\|\lambda f_{|I \cap J}\| = \|g_{|I \cap J}\| \leq \|g\|$ by Lemma 2.5. Thus $\|\lambda q\| \geq |\lambda| \, \|q\|$.

If $(I_1,f_1) \in q_1$ and $(I_2,f_2) \in q_2$ then

$$\|f_1 + f_2\| = \sup\{ \|f_1(x) + f_2(x)\| \mid x \in I_1 \cap I_2, \ \|x\| = 1\}$$

$$\leq \sup\{ \|f_1(x)\| \mid x \in I_1, \ \|x\| = 1\} + \sup\{ \|f_2(x)\| \mid x \in I_2, \ \|x\| = 1\}$$

$$= \|f_1\| + \|f_2\|$$

and

$$\|f_1 f_2\| = \sup\{ \|f_1(f_2(x))\| \mid x \in I_2 I_1, \ \|x\| = 1\}$$

$$\leq \|f_1\| \sup\{ \|f_2(x)\| \mid x \in I_2, \ \|x\| = 1\}$$

$$= \|f_1\| \, \|f_2\|.$$

Therefore

$$\inf\{ \|f\| \mid (I,f) \in q_1 + q_2\}$$

$$\leq \inf\{ \|f_1 + f_2\| \mid (I_1,f_1) \in q_1, \ (I_2,f_2) \in q_2\}$$

$$\leq \inf\{ \|f_1\| + \|f_2\| \mid (I_1,f_1) \in q_1, \ (I_2,f_2) \in q_2\}$$

$$= \inf\{ \|f_1\| \mid (I_1,f_1) \in q_1\} + \inf\{ \|f_2\| \mid (I_2,f_2) \in q_2\}$$

which implies $\|q_1 + q_2\| \leq \|q_1\| + \|q_2\|$.

Similarly

$$\inf\{ \|f\| \mid (I,f) \in q_1 q_2\}$$

$$\leq \inf\{ \|f_1 f_2\| \mid (I_1,f_1) \in q_1, \ (I_2,f_2) \in q_2\}$$

$$\leq \inf\{ \|f_1\| \, \|f_2\| \mid (I_1,f_1) \in q_1, \ (I_2,f_2) \in q_2\}$$

$$= \inf\{ \|f_1\| \mid (I_1,f_1) \in q_1\} \cdot \inf\{ \|f_2\| \mid (I_2,f_2) \in q_2\}$$

which shows $\|q_1 q_2\| \leq \|q_1\| \cdot \|q_2\|$.

If $\|q\| = 0$ then for each $k \in \mathbb{N}$ there is $(I_k,f_k) \in q$ such that $\|f_k\| < \frac{1}{k}$. Since $I_k \subseteq I_{(q)}$ and $f_k = f_{(q)|I_k}$ this

yields a sequence $(x_k)_{k\in\mathbb{N}}$ in $I_{(q)}$ with $\|x_k\| = 1$ for all k such that $\lim_{k\to\infty} f_{(q)}(x_k) = 0$. Therefore a necessary condition for $\|q\| = 0$ is that $f_{(q)}$ is not bounded below.

If $q \in Z(Q(A))$, $q \neq 0$ then $f_{(q)}$ is a non-zero A-bimodule homomorphism (Proposition 2.6) and therefore bounded below on $I_{(q)}$ (Lemma 2.3). Thus, $\|\cdot\|$ is a norm on $Z(Q(A))$.

Let now $I \in \mathcal{I}$ and take $x \in I$ with $\|x\| = 1$. By Lemma 1.1 there is $c_M > 0$ such that for all $a \in A$

$$c_M\|a\| \leq \|M_{a,x}\| = \sup\{ \|azx\| \mid z \in A, \|z\| = 1\}$$

$$\leq \sup\{ \|ay\| \mid y \in I, \|y\| \leq 1\} = \|L_{a|I}\|.$$

Therefore, $c_M\|a\| \leq \inf\{ \|L_{a|I}\| \mid I \in \mathcal{I} \}$

$$= \inf\{ \|f\| \mid (I,f) \in [(A,L_a)]\}.$$

Thus, $\|\cdot\|$ is also a norm on A.□

Corollary 2.13. *The extended centroid of an ultraprime normed algebra is isomorphic to the complex numbers.*

Proof. By 2.6 and 2.12, $Z(Q(A))$ is a normed field if A is an ultraprime normed algebra, thus $Z(Q(A)) = \mathbb{C}1$ by Mazur's theorem.□

Definition 2.14. Let $N = \{ q \in Q(A) \mid \|q\| = 0\}$. By Lemma 2.12, N is an ideal in $Q(A)$. The quotient $Q_A := Q(A)/N$ endowed with the induced norm is called *the normed algebra of quotients* of the ultraprime algebra A.

Theorem 2.15. *The normed algebra of quotients Q_A of an ultraprime normed algebra A is a unital ultraprime normed algebra containing a homeomorphic image of* A.

Proof. Evidently, Q_A is a unital normed algebra under the norm $\|q + N\| := \|q\|$. If $a \in A$ then

$$c_M\|a\| \leq \|[(A,L_a)] + N\| \leq \|a\|$$

by 2.12, hence A can be considered as a closed subalgebra of Q_A. In the following we will specify the norm of A by $\|\cdot\|_A$. Let $p,q \in Q(A)$ with $\|p\| = \|q\| = 1$. By definition of the norm there is $(I,f) \in p$ with $\|f\| \geq 1$. Thus, for every $0 < \varepsilon < 1$, there is $x \in I$, $\|x\|_A = 1$ such that $\|f(x)\|_A \geq 1 - \varepsilon$. Equivalently, there is $x \in I$, $1 \geq \|x\| \geq c_M$ such that $\|px\|_A \geq 1 - \varepsilon$ (observe that $f(x) = px \in A$).

Similarly, for every $0 < \varepsilon < 1$ there is $y \in A$, $1 \geq \|y\| \geq c_M$ with $\|qy\|_A \geq 1 - \varepsilon$.

By Lemma 1.1, $\|M_{px,qy}\| \geq c_M\|px\|_A\,\|qy\|_A \geq c_M(1 - \varepsilon)^2$. Therefore, there exists $z \in A$ such that $\|z\|_A = 1$ and $\|pxzqy\|_A \geq c_M(1 - \varepsilon)^2 - \varepsilon$. Since $\|xz\|_A \leq 1$, i.e. $c_M \leq \|xz\| \leq 1$, and

$$\|M_{p,q}xz\| \geq \|pxzq\|\,\|y\| \geq c_M\|pxzqy\|_A \geq c_M(c_M(1 - \varepsilon)^2 - \varepsilon)$$

$$= c_M^2 - 2c_M^2\varepsilon - c_M\varepsilon + c_M^2\varepsilon^2,$$

it follows that $\|M_{p,q}\| \geq c_M^2$.

By means of this, for every $p + N$, $q + N \in Q_A$

$$\|M_{p+N,q+N}\| = \|M_{p,q}\| \geq c_M^2\|p + N\|\,\|q + N\|.$$

By Lemma 1.1, Q_A is therefore ultraprime.□

Remarks 2.16. 1. For every ultraprime normed algebra A we denote by $\tilde{Q}_A$ the completion of Q_A and call it the *Banach algebra of quotients of* A. By Proposition 1.6, $\tilde{Q}_A$ is an ultraprime Banach algebra.

2. Since $Z(Q(A)) \subseteq Z(Q_A)$, Corollary 2.13 also follows directly from Theorem 2.15 without using Mazur's theorem.

We conclude this chapter with some examples of Q_A.

Examples 2.17. If A is a simple unital algebra then $\mathcal{I} = \mathcal{I}_c = \{A\}$ and thus $Q_a(A) = Q_c(A) = Q(A) = Q_A = A$. More generally, if A is a topologically simple Banach algebra, e.g. the compact operators on a Hilbert space, then $Q(A) = Q_c(A) = \{f:A \to A \mid f$ is a continuous right A-module homomorphism$\} = Q_A$, since $\|f\| = \|f_{|I}\|$ for any $I \in \mathcal{I}$, and coincides with the algebra of all left centralizers of A. In the special case $A = K(H)$ we obtain $Q_A = L(H)$. (Also, $Q_a(A) = \mathrm{End}_A(F(H))$, the ring of all right A-module homomorphisms from the finite rank operators into itself, cf. [126], ex. XII.2.3.)

Let now $A = L(H)$. If $f:I \to A$ is a continuous right A-module homomorphism on a closed non-zero ideal I of A and $\{e_\lambda\}$ is an approximate identity in I, $\{f(e_\lambda)\}$ has a weak cluster point $a_f \in A$. By passing to a subnet we may assume that $f(e_\lambda)x$ converges weakly to $a_f x$ for every $x \in I$. On the other hand, $f(e_\lambda)x = f(e_\lambda x)$ tends to $f(x)$ in norm, hence $f(x) = a_f x$ for all $x \in I$. From this and the fact that every non-zero ideal of A is ultraweakly dense in A, we deduce $Q(A) = Q_c(A) = Q_A = A$. Clearly, the same argument applies to any factor (von Neumann algebra with trivial center).

The ring of quotients of an arbitrary ultraprime normed algebra A can also be considered as a direct limit of complex vector spaces. To this end we take the directed set $\mathcal{I}$ endowed with the partial order $I_1 \leq I_2$ iff $I_2 \subseteq I_1$. A coherent family of linear mappings $\mathrm{Hom}_A(I_1,A) \to \mathrm{Hom}_A(I_2,A)$ where $I_1 \leq I_2$ is then given by restricting the continuous right A-module homomorphisms from I_1 into A to I_2. Then $Q(A) = \varinjlim \mathrm{Hom}_A(I,A)$.

If A is a prime C*-algebra and $LM(I)$ is the space of all left multipliers of the closed non-zero ideal I, then $Q(A) = Q_c(A) = \varinjlim LM(I)$ (use 2.2 and 2.7).

Clearly, the results in this chapter can only be a first step in a more detailed investigation of normed algebras of quotients. A comprehensive account of the algebraic theory of rings of quotients is given in [126]; e.g. a generalization of the direct limit formula can be found in [126], ch. IX. We shall use Q_A in Chapter 4 to obtain a fundamental result on elementary operators.

3. BASIC PROPERTIES OF PRIME C*-ALGEBRAS

Prime C*-algebras constitute, as we shall see, the most important class of ultraprime Banach algebras, and they play the same rôle in these algebras as Hilbert spaces among Banach spaces. Thus many results become smoother and can be obtained more easily. This is mainly a consequence of the intimate relations between algebraic and topological structures in a C*-algebra.

First of all we compare 'primeness' with other conditions on the (ideal) structure of a C*-algebra A. There are always the following implications.

(i) A *is simple* => (ii) A *is primitive* => (iii) A *has a faithful factor representation* => (iv) A *is prime* => (v) A *has trivial center.*

It is well known that for W*-algebras (v) => (ii) also holds. The algebras $A = L(H)$ and $A = K(H) \oplus K(H)$ where H is an infinite dimensional Hilbert space may serve as counterexamples for (ii) => (i) and (v) => (iv), respectively.
There are several other conditions which ensure the validity of (iv) => (ii):

— A is separable ([116], 4.3.6)

— A is not antiliminal ([9], lem. 3.1; see also 6.14.2)

— A contains a non-zero closed ideal which is a primitive C*-algebra ([8], p. 678)

— A contains minimal projections (this will be exploited in Chapter 6).

It is a long standing problem raised by I.Kaplansky whether any prime C*-algebra has to be primitive.

A characterization of prime C*-algebras in terms of the state space $S(A)$ was given by Archbold ([9], thm 3.3) as follows.

> *A C*-algebra* A *is prime if and only if the set of factorial states is weak* dense in* $S(A)$.

Together with the results in [10] (see also [11]) this extends the earlier theorem by Glimm, Takesaki and Tomiyama ([65], [137]) that the set of pure states of a C*-algebra A is weak* dense in $S(A)$ if and only if A is prime and either anti-liminal or one-dimensional.

Our first result in this chapter both gives the connection to the previous chapters and a first taste of the nice properties of elementary operators on prime C*-algebras. For the notation see Appendix 2 and p.50.

Proposition 3.1. *Let* A *be a unital C*-algebra. Then* A *is prime if and only if for all* $a,b \in A$ *any (and hence each) of the following conditions is satisfied.*

(a) $\|M_{a,b}\| = \|a\| \, \|b\|$;

(b) $\sigma(M_{a,b}) = \sigma(a) \, \sigma(b)$;

(c) $\sigma_\ell(M_{a,b}) = \sigma_\ell(a) \, \sigma_r(b)$;

(c') $AP\sigma(M_{a,b}) = AP\sigma(a) \, AP\sigma_r(b)$;

(d) $\sigma_r(M_{a,b}) = \sigma_r(a) \, \sigma_\ell(b)$;

(d') $AP\sigma_r(M_{a,b}) = AP\sigma_r(a) \, AP\sigma(b)$;

(e) $r(M_{a,b}) = r(a) \, r(b)$.

Proof. It is clear from the definition that (a) implies the primeness of A. For the converse, assume A to be prime and

take $a,b \in A$ with $\|a\| = \|b\| = 1$. Then $\|M_{a,b}\| \leq 1$. First, suppose $a,b \in A_+$. For $0 < \varepsilon < 1$ the functional calculus yields non-zero elements $c,d \in A_+$ such that $0 < (1 - \varepsilon)c \leq ac$ and $0 < (1 - \varepsilon)d \leq db$. Choosing $u \in A$ with $x := cud \neq 0$ we obtain

$$\begin{aligned}\|M_{a,b}x\|^2 &= \|acudb\|^2 = \|bdu^*ca^2cudb\| \geq (1 - \varepsilon)^2\|bdu^*c^2udb\| \\ &= (1 - \varepsilon)^2\|cudb\|^2 = (1 - \varepsilon)^2\|cudb^2du^*c\| \\ &\geq (1 - \varepsilon)^4\|cud^2u^*c\| = (1 - \varepsilon)^4\|cud\|^2 \\ &= (1 - \varepsilon)^4\|x\|^2.\end{aligned}$$

Consequently, $\|M_{a,b}\| \geq 1$.

If $a,b \in A$ are arbitrary, the equality

$$\begin{aligned}\|a\|^2\|b\|^2 &= \|a^*a\|\,\|bb^*\| = \|M_{a^*a,bb^*}\| \leq \|M_{a^*,b^*}\|\,\|M_{a,b}\| \\ &\leq \|a\|\,\|b\|\,\|M_{a,b}\|\end{aligned}$$

completes the argument.

The spectral identities will turn out to be special cases of Theorem 7.8; hence we only sketch the proof. The equivalences (a) <=> (b) <=> (c') <=> (e) were proved in [109], prop. 4.5; (a) <=> (d') follows from a similar argument. Finally, (c) <=> (c') and (d) <=> (d') are consequences of the fact that $\sigma_\ell(x) = AP\sigma(x)$ and $\sigma_r(x) = AP\sigma_r(x)$ for all $x \in A$ (see A.2.6), and hence that $\sigma_\ell(M_{a,b}) = AP\sigma(M_{a,b})$ and $\sigma_r(M_{a,b}) = AP\sigma_r(M_{a,b})$. □

In [109], cor. 4.9, we calculated the norm of $M_{a,b}$ on arbitrary C*-algebras, and proved 3.1.(a) under the assumption that A is a prime W*-algebra (ibid., lem. 4.7). The fact that this proof can be modified for prime C*-algebras was pointed out to us by B.Blackadar (private communication).

The most useful condition in 3.1 is condition (a) (which does not require A to be unital). Together with Lemma 1.1 it yields immediately the following fundamental result.

Corollary 3.2. *A C*-algebra is prime if and only if it is ultraprime.*

Next we develop some more possibilities of making a prime C*-algebra larger or smaller without loss of primeness.

Lemma 3.3. *A C*-algebra without identity is prime if and only if its unitization is prime.*

This holds for any ring (see e.g. [8], lem. 3.4).

It also follows from ring theory that any closed ideal of a prime C*-algebra is a prime C*-algebra itself. However, as was pointed out to us by B.Magajna (private communication), more is true.

Lemma 3.4. *Let* B *be a hereditary C*-subalgebra of the prime C*-algebra* A. *Then* B *is prime.*

Proof. Let $a,b \in B$, $x \in A$ and put $L(B_+) = \{z \in A \mid z^*z \in B_+\}$. If $y_x = a^*xb^*$ it follows from

$$0 \leq y_x^* y_x = bx^*aa^*xb^* \leq \|a\|^2\|x\|^2 bb^*$$

that $y_x \in L(B_+)$, and similarly $y_x^* \in L(B_+)$. Thus y_x belongs to $L(B_+) \cap L(B_+)^* = B$.

Now if $ayb = 0$ for all $y \in B$, then $aa^*xb^*b = 0$ for all $x \in A$. From the primeness of A we conclude that $aa^* = 0$ or $b^*b = 0$. Hence, B is prime.□

The next lemma may be regarded as a supplement to the following theorem due to Olsen [114] and Kato [95].

Let A *be a von Neumann algebra and let* I *be a norm closed ideal of* A. *If* $a,b \in A$ *are such that* $ab \in I$ *then there exists a projection* p *in* A *with* $ap \in I$ *and* $(1 - p)b \in I$.

Lemma 3.5. *Let* A *be a factor. Then any closed ideal of* A *is a prime ideal.*

Proof. For a closed ideal I in A denote by π the canonical quotient map from A onto $A_{/I}$. Take $a,b \in A$ with $\pi(a) \neq 0$ and $\pi(b) \neq 0$. Without loss of generality we may assume $a,b \geq 0$. By the Spectral Theorem there are projections $p,q \in A$ and $\varepsilon > 0$ such that $\pi(p) \neq 0$, $\pi(q) \neq 0$ and $ap \geq \varepsilon p$, $qb \geq \varepsilon q$. Indeed, if $a = \int_0^\infty \lambda \, de_\lambda$ is the spectral decomposition of a, then, for $\varepsilon > 0$ and $p_\varepsilon = 1 - e[0,\varepsilon)$, we have $ap_\varepsilon \geq \varepsilon p_\varepsilon$ and $\|a - ap_\varepsilon\| \leq \varepsilon$. If $p_\varepsilon \in I$ for all $\varepsilon > 0$ then $a \in I$, a contradiction.

By the Comparability Theorem we obtain a partial isometry $u \in A$ such that, without restriction, $p \geq uu^*$ and $u^*u = q$. Therefore

$$\begin{aligned} \|\pi(a)\pi(u)\pi(b)\|^2 &= \|\pi(apuqb)\|^2 = \|\pi(bqu^*pa^2puqb)\| \\ &\geq \varepsilon^2 \|\pi(bqu^*puqb)\| = \varepsilon^2 \|\pi(uqb)\|^2 \\ &= \varepsilon^2 \|\pi(uqb^2qu^*)\| \geq \varepsilon^4 \|\pi(uqu^*)\| \\ &= \varepsilon^4 \|\pi(u)\|^2 = \varepsilon^4. \end{aligned}$$

Hence $aub \notin I$, i.e. I is a prime ideal.□

A different argument for 3.5 stems from the fact that the set of closed ideals of a factor is totally ordered under inclusion ([136], exer. V.1.5).

The multiplier algebra M(A) of a C*-algebra A is defined as the idealizer of A in its enveloping W*-algebra

denoted by A^{**}. Recall two other descriptions of $M(A)$ ([116], 3.12). An element $x \in A^{**}$ belongs to $M(A)$ iff its left and right regular representations, L_x and R_x, leave A invariant. Hence, (R_x, L_x) defines a double centralizer of A. On the other hand, any double centralizer (ρ_r, ρ_ℓ) of A (i.e. any pair of linear mappings ρ_r, ρ_ℓ on A such that $\rho_r(uv) = u\rho_r(v)$, $\rho_\ell(uv) = \rho_\ell(u)v$ and $\rho_r(u)v = u\rho_\ell(v)$ for all $u,v \in A$) gives rise to an element $x \in A^{**}$ such that $\rho_r = R_x$ and $\rho_\ell = L_x$.

Lemma 3.6. *For every C*-algebra* A *the following conditions are equivalent:*

(a) A *is prime.*

(b) $M(A)$ *is prime.*

Proof. Since A is a closed ideal in $M(A)$ the implication (b) => (a) follows. To prove the other direction we take $a,b \in M(A)$ such that $b \neq 0$ and $axb = 0$ for all $x \in M(A)$. Then $auavbwb = 0$ for all $u,v,w \in A$. There exists an element $w \in A$ with $bwb \neq 0$. Otherwise, $bwb = 0$ for all $w \in A$, hence for all $w \in A^{**}$ which contradicts $b \neq 0$. Since $M_{aua,bwb} = 0$ on A for all $u \in A$ and A is prime, we obtain $aua = 0$ for all $u \in A$, hence for all $u \in A^{**}$. Thus $a = 0$. □

The next result determines the bimodule homomorphisms from a not necessarily closed ideal into a prime C*-algebra. It will provide a proof of Theorem 4.2 for C*-algebras without reference to the construction of Chapter 2.

Proposition 3.7. *If* I *is an ideal in a prime C*-algebra* A, *then every* A*-bimodule homomorphism* $f: I \to A$ *is a multiplication by a complex number.*

Proof. By Lemma 2.3 and Corollary 3.2, f is continuous. Hence, we can extend f to the closure $\overline{I}$ of I (and denote the extension again by f). Since every $x \in \overline{I}$ can be written as $x = u|x|^{1/2}$ for some $u \in \overline{I}$ ([116], 1.4.6), $f(x) = f(u)|x|^{1/2}$ belongs to $\overline{I}$ and thus $f(\overline{I}) \subseteq \overline{I}$. Therefore, (f,f) defines a double centralizer of $\overline{I}$ corresponding to a multiplier c in $M(\overline{I})$. From $(cx - xc)y = f(x)y - xf(y) = 0$ for all $x,y \in \overline{I}$ we conclude $c \in Z(M(\overline{I}))$, the center of the multiplier algebra of $\overline{I}$. But, since $\overline{I}$ is prime, $M(\overline{I})$ is prime by 3.6. Hence, $Z(M(\overline{I})) = \mathbb{C}1$, and f is a multiplication by a complex number. □

We conclude this chapter with some remarks on the order structure of a prime C*-algebra. It is well known that the self-adjoint part A_h of a C*-algebra A is a lattice under the usual ordering if and only if A is commutative ([122]). On the other hand, a C*-algebra A is said to be an *antilattice* ([92], p. 505) if, whenever two elements in A_h have an infimum, they are comparable. Archbold ([8], thm 3.7) characterized the antilattices as those C*-algebras which are prime, while Kadison ([92], cor. 11 and thm 12) had done the same before for W*-algebras. In [67], Green gave a much simpler proof of Archbold's result. Using a more geometric approach, we now add another direct proof. After we had obtained our proof and communicated it informally, we learned that Chu proved in [34], thm 1.3 a more general theorem stating that a compact convex set K is prime if and only if the partially ordered Banach space $A(K)$ of all continuous affine real-valued functions on K is an antilattice (see also [13], 3.5.1). From this theorem which extended the earlier result for simplexes by Effros and Kazdan ([4], II.7.15) Archbold's theorem can be derived easily by using Effros' duality ([116], 3.10.8) between

closed ideals and invariant faces of the state space of a C*-algebra (cf. [13], 5.4.5; see also [33], thm 3.5). Though both in Chu's proof as well as in ours the main tool is Hervé's characterization of extreme points (see below), we believe that there is still some interest in our approach. First, we don't need any additional information like in [8], lem. 3.4 and 3.6 to derive the non-unital from the unital case. But more important, we replace the relations between prime ideals and prime faces by a simple application of the Cauchy-Schwarz inequality.

The key is the following

Lemma 3.8. *The following conditions on two positive elements* a,b *of a C*-algebra* A *are equivalent:*

(a) $\inf(a,b) = 0$, (b) $M_{a,b} = 0$, (c) $I_a I_b = 0$.

(Here, I_x *denotes the closed ideal generated by* $x \in A$.*)*

Proof. The essential step is (a) => (b) for which we give two short proofs each of which has some interesting aspects. The first is of a more algebraic nature and is a slight modification of Green's argument ([67], thm 2.2).

For $a,b \in A_+$ and $x \in A$ define $w_x \in A_h$ by

$$w_x = a^{1/2}xx^*a^{1/2} + b^{1/2}x^*xb^{1/2} + 2\mathrm{Re}(a^{1/2}xb^{1/2}),$$

where $\mathrm{Re}(y)$ denotes the real part of $y \in A$. Then $w_x \geq \mathrm{Re}(a^{1/2}xb^{1/2})$, and $-w_x \leq a, b$ since

$$a + w_x = (a^{1/2} + xb^{1/2})^*(a^{1/2} + xb^{1/2}) + a^{1/2}xx^*a^{1/2} \geq 0$$

and

$$b + w_x = (a^{1/2}x + b^{1/2})(a^{1/2}x + b^{1/2})^* + b^{1/2}x^*xb^{1/2} \geq 0.$$

If $M_{a,b} \neq 0$, take $u \in A$ and $\phi \in A_+^*$ (the positive cone in the dual A^*) with $\langle \phi, \mathrm{Re}(a^{1/2}ub^{1/2}) \rangle \neq 0$. We now claim that

$w_x \not\geq 0$ for some $x \in A$. In fact, if we put

$$\lambda = \langle \phi, a^{1/2} u u^* a^{1/2} + b^{1/2} u^* u b^{1/2} \rangle, \quad \mu = \langle \phi, a^{1/2} u b^{1/2} \rangle \quad \text{and}$$

$$x = -\frac{\bar{\mu}}{1+\lambda}\, u$$

then

$$\langle \phi, w_x \rangle = \frac{|\mu|^2 \lambda}{(1+\lambda)^2} - 2\,\frac{|\mu|^2}{1+\lambda} = -\,\frac{|\mu|^2 (2+\lambda)}{(1+\lambda)^2} < 0.$$

But, if $\inf(a,b) = 0$, then $-w_x \leq 0$ for all $x \in A$, hence $w_x \geq 0$ for all $x \in A$.
Thus, 0 is not the infimum of a and b.

The second proof for (a) => (b) is more geometrically and uses Kadison's function representation (cf. [116], 3.10). We may identify, as partially ordered real Banach spaces, A_h and $A_o(\mathcal{Q})$, the space of all continuous affine real-valued functions on the quasi-state space $\mathcal{Q}$ of A which vanish at zero. Hence, if $\inf(a,b) = 0$ then, for each pure state ϕ on A, $\langle \phi, a \rangle = 0$ or $\langle \phi, b \rangle = 0$. Indeed, by a well known result of Hervé ([4], I.4.1), $\inf(a,b)(\phi) = \inf\{a(\phi), b(\phi)\}$ for every extreme point ϕ of $\mathcal{Q}$. By the Cauchy-Schwarz inequality we obtain

$$|\langle \phi, a^{1/2} x b^{1/2} \rangle|^2 \leq \langle \phi, a \rangle \langle \phi, b^{1/2} x^* x b^{1/2} \rangle \leq \|x\|^2 \langle \phi, a \rangle \langle \phi, b \rangle = 0$$

for each pure state ϕ and each $x \in A$.
Thus, $a^{1/2} x b^{1/2} = 0$ for each $x \in A$ which yields $M_{a,b} = 0$.

(b) <=> (c) Since $I_a = \overline{AaA}$ and $I_b = \overline{AbA}$ we have

$$I_a I_b \subseteq \overline{A(M_{a,b})A} \subseteq I_a \cap I_b = I_a I_b$$

([35], prop. 5.1) from which the assertion follows.

(b) => (a) If $\inf(a,b) \neq 0$ then there is $c \in A_h$ with $c \leq a$, b and $c \not\leq 0$. Take a pure state ϕ such that $0 < \langle \phi, c \rangle \leq \langle \phi, a \rangle$ and $\langle \phi, c \rangle \leq \langle \phi, b \rangle$. Then $\pi_\phi(a) \neq 0$ and

$\pi_\phi(b) \neq 0$ where π_ϕ denotes the corresponding irreducible representation of A. Since primitive algebras are prime, $M_{\pi_\phi(a),\pi_\phi(b)} \neq 0$ thus $M_{a,b} \neq 0$.□

Corollary 3.9 (thm 3.7 of [8])

A C-algebra is prime if and only if it is an antilattice.*

Proof. If A is a prime C*-algebra and $a,b \in A_h$ have an infimum $c \in A_h$, then $a_1 = a - c$, $b_1 = b - c$ are positive with infimum 0. By Lemma 3.8 and the primeness of A we obtain $a_1 = 0$ or $b_1 = 0$, i.e. $a = c \leq b$ or $b = c \leq a$. Thus , A is an antilattice.

Conversely, suppose A is an antilattice and take $a,b \in A$ with $M_{a,b} = 0$. Since $M_{a^*a,bb^*} = 0$ we conclude from Lemma 3.8 that $a^*a = 0$ or $bb^* = 0$. Thus, A is prime. □

Remarks. 1. Archbold's basic result ([8], thm 3.1) from which he deduced the above characterization of prime C*-algebras also follows at once from Lemma 3.8 (see [67], cor 2.3). It is perhaps of interest to observe that though Hervé's result was published in 1961, Kadison already knew (but did not use) it when he proved Corollary 3.9 for W*-algebras ([4], Notes on pp. 43 and 79; [93], proof of thm 4.1).

2. Since a hereditary C*-subalgebra of an antilattice is clearly an antilattice, Corollary 3.9 provides an alternative proof of Lemma 3.4.

4. THE IDENTITY $S = 0$

We begin this chapter with a formal definition of elementary operators.

Definition 4.1. Let A be an algebra. If $a = (a_1,\ldots,a_n)$ and $b = (b_1,\ldots,b_n)$ belong to A^n then the linear mapping $S:A \to A$, $x \mapsto \sum_{j=1}^{n} a_j x b_j$ is called an *elementary operator on* A. The set of all elementary operators on A will be denoted by $\mathcal{E}\ell(A)$; it is the algebra generated by the semigroup $M(A)$ of all multiplications on A. Clearly, if A is normed, then $\mathcal{E}\ell(A) \subseteq L(A)$.

By this definition, every elementary operator has a representation $S = \sum_{j=1}^{n} M_{a_j,b_j}$, and in general there is no uniqueness in such a representation. In this chapter we give a criterion for uniqueness in terms of linear independency of the multiplying elements a_j and b_j if the algebra A is ultraprime. Evidently, it is enough to study the case $S = 0$. In the following two chapters we shall exploit this result to obtain characterizations of completely positive and of (weakly) compact elementary operators on prime C*-algebras, respectively.

In spite of its obvious significance, the question of the uniqueness of representation of an elementary operator seems to have been neglected until 1979 when Fong and Sourour gave an answer for the algebra $L(E)$ of all bounded operators on a Banach space E and for the Calkin algebra on a separable Hilbert space ([64], thms 1 and 3). In the proof of the first case, they used rank-one operators while for the Calkin algebra

their main tool was Voiculescu's non-commutative version of the Weyl-von Neumann Theorem. More recently, the question was taken up again by Synnatzschke [132] who extended the technique of Fong and Sourour to algebras with many one-dimensional elements. (In our terms, he assumed that the multiplication $M_{a,b}$ does not vanish already on the set of one-dimensional elements whenever a and b are non-zero. It follows that these algebras are primitive and contain many minimal idempotents.) However, his approach does not cover the Calkin algebra case.

We now state the main result of this chapter.

Theorem 4.2. *Let* A *be an ultraprime normed algebra, let* $a,b \in A^n$ *and let* $S = \sum_{j=1}^{n} M_{a_j,b_j}$ *be the corresponding elementary operator on* A. *Assume that* $\{b_1,\dots,b_m\}$ *is a maximal linearly independent subset of* $\{b_1,\dots,b_n\}$ *and that* λ_{jk} *are complex numbers such that* $b_j = \sum_{k=1}^{m} \lambda_{jk} b_k$ *for* $m+1 \le j \le n$.

Then the following are equivalent:

(a) $S = 0$

(b) $a_k = -\sum_{j=m+1}^{n} \lambda_{jk} a_j$ *for* $1 \le k \le m$.

In case $m = n$, *condition* (b) *should be understood as* $a_k = 0$ *for all* k.

We note that an algebra for which 4.2 holds has necessarily to be prime (consider the case $n = 1$) and has to have trivial center (if $\dim_{\mathbb{C}} Z(A) \ge 2$ then there are linearly independent elements $b_1, b_2 \in Z(A)$ for which $M_{b_2,b_1} + M_{-b_1,b_2} = 0$).

Proof. Evidently (b) is sufficient for (a) if $m = n$.

The identity

$$S = \sum_{j=1}^{m} M_{a_j,b_j} + \sum_{j=m+1}^{n} M_{a_j,b_j}$$

$$= \sum_{j=1}^{m} M_{a_j,b_j} + \sum_{j=m+1}^{n} M_{a_j,\sum_{k=1}^{m} \lambda_{jk} b_k}$$

$$= \sum_{k=1}^{m} M_{a_k,b_k} + \sum_{k=1}^{m} M_{\sum_{j=m+1}^{n} \lambda_{jk} a_j,b_k}$$

$$= \sum_{k=1}^{m} M_{a_k + \sum_{j=m+1}^{n} \lambda_{jk} a_j,b_k}$$

shows the sufficiency also if $m < n$; moreover, in the proof of necessity we may assume $m = n$.

The argument is by induction on n. If $n = 1$, the assertion follows from the definition of primeness. If $n > 1$ and there exists some $j \in \{1,\ldots,n\}$ such that $a_j \neq 0$, by the induction hypothesis we may assume $a_j \neq 0$ for all j . Putting $I_j = Aa_jA$ for $1 \leq j \leq n$ we obtain non-zero ideals in A . Define $f_j: I_1 \to A$ by $\sum_i x_i a_1 y_i \mapsto \sum_i x_i a_j y_i$ for any finite subsets $\{x_i\},\{y_i\}$ of A. In order to prove that f_j is a well-defined A-bimodule homomorphism we assume $\sum_i x_i a_1 y_i = 0$. Since $S = 0$, we have $\sum_{j=1}^{n} a_j y_i z b_j = 0$ for all $z \in A$. Therefore

$$0 = \sum_i x_i a_1 y_i z b_1 = -\sum_{j=2}^{n} \sum_i x_i a_j y_i z b_j$$

for all $z \in A$. By the induction hypothesis $\sum_i x_i a_j y_i = 0$ for all $2 \leq j \leq n$. Thus f_j is well-defined.
By Corollary 2.13 there are complex numbers γ_j such that $f_j(x) = \gamma_j x$ for all $x \in I_1$, which are all non-zero since f_j maps I_1 onto I_j.

Moreover

$$y(\gamma_j a_1 - a_j)z = \gamma_j y a_1 z - y a_j z = f_j(y a_1 z) - y a_j z = 0$$

for all $y,z \in A$ from which $\gamma_j a_1 = a_j$ follows. Finally,

$$M_{a_1, \sum_{j=1}^{n} \gamma_j b_j} = \sum_{j=1}^{n} M_{\gamma_j a_1, b_j} = \sum_{j=1}^{n} M_{a_j, b_j} = 0$$

and the primeness of A imply $\sum_{j=1}^{n} \gamma_j b_j = 0$. This contradiction completes the proof. □

If A is a prime C*-algebra we may replace Corollary 2.13 in the above proof by the more direct Proposition 3.7. Since $\mathcal{L}(E)$ is ultraprime, and the Calkin algebra on a separable Hilbert space is a simple C*-algebra ([23]), we recover the results by Fong and Sourour. Observe that by Lemma 3.5, Theorem 4.2 is also applicable to the Calkin algebra on a non-separable Hilbert space. We would like to mention that the use of the A-bimodule homomorphisms f_j was suggested by a similar result in [82], lem.1.3.2.

Remark. Using more elaborate C*-theory it is possible to deduce Theorem 4.2 for prime C*-algebras from the result by Fong and Sourour. Once having proved 4.2 for $A = \mathcal{L}(H)$, it immediately extends to primitive C*-algebras. Now, if A is an arbitrary prime C*-algebra and B_o is a separable C*-subalgebra of A containing $a_1,\ldots,a_n,b_1,\ldots,b_n$ then, as was pointed out by L.Zsidó (private communication), one can apply the result in [50], prop. 3.1 to obtain a separable prime C*-subalgebra B of A which contains B_o. Since $S = 0$ on B and B is primitive ([116], 4.3.6), we arrive at the assertion.

A more symmetric formulation of Theorem 4.2 is obtained in the first corollary.

Corollary 4.3. *Let* A *be an ultraprime normed algebra. If* $\{a_1,\dots,a_n\}$ *and* $\{b_1,\dots,b_n\}$ *are linearly independent subsets of* A, *then* $\{M_{a_1,b_1},\dots,M_{a_n,b_n}\}$ *is a linearly independent subset of* $L(A)$.

Proof. Let $\sum_{j=1}^{n} \lambda_j M_{a_j,b_j} = 0$ for some $\lambda_j \in \mathbb{C}$. By 4.2, $\lambda_j a_j = 0$ for all j. Since $\{a_1,\dots,a_n\}$ is also linearly independent we must have $\lambda_j = 0$. □

As another immediate consequence we obtain that on a finite dimensional prime algebra every (bounded) linear operator is elementary.

Corollary 4.4. *Let* A *be a finite dimensional prime algebra. Then* $E\ell(A) = L(A)$.

Proof. If $\dim A = n < \infty$ and $\{b_1,\dots,b_n\}$ is a basis of A, then, since A is ultraprime (cf. Chapter 1), the set $\{M_{b_k,b_\ell} \mid 1 \le k,\ell \le n\}$ is linearly independent: If

$$0 = \sum_{k,\ell=1}^{n} \lambda_{k\ell} M_{b_k,b_\ell} = \sum_{\ell=1}^{n} M_{\sum_{k=1}^{n} \lambda_{k\ell} b_k, b_\ell}$$

for some $\lambda_{k\ell} \in \mathbb{C}$, we have $\sum_{k=1}^{n} \lambda_{k\ell} b_k = 0$ for all ℓ, thus $\lambda_{k\ell} = 0$ for all k and ℓ.
Since $\dim L(A) = n^2$, we obtain a basis of $L(A)$ contained in $E\ell(A)$. □

Generalizing this we establish a relation between the algebra of all elementary operators on A and the tensor product of A 'with itself'. To this end, denote by $^{\circ}A$ the *reversed*

algebra of the normed algebra A, i.e. the antiisomorphic algebra obtained by introducing the product $x \circ y = yx$ on A. Suppressing the antiisomorphism we define a continuous bilinear multiplicative mapping ${}^{\circ}M$ from $A \times {}^{\circ}A$ into $L(A)$ by ${}^{\circ}M(a,b) = M_{a,b}$. Then there is a unique homomorphism θ from the algebraic tensor product $A \otimes {}^{\circ}A$ into $L(A)$ such that $\theta(a \otimes b) = {}^{\circ}M(a,b)$; clearly, $\theta(A \otimes {}^{\circ}A) = El(A)$. We call θ *the canonical homomorphism from* $A \otimes {}^{\circ}A$ *onto* $El(A)$.
If θ is injective and $\theta(a \otimes b) = M_{a,b} = 0$ for some $a,b \in A$ then $a \otimes b = 0$, thus $a = 0$ or $b = 0$. Hence, A is prime.

Corollary 4.5. *Let* A *be a normed algebra. If* A *is ultraprime then the canonical homomorphism* $\theta: A \otimes {}^{\circ}A \to El(A)$ *is injective.*

Proof. Suppose that $\theta(u) = 0$ for some $u \in A \otimes {}^{\circ}A$. As we may choose a representation $u = \sum_{j=1}^{n} a_j \otimes b_j$ with $\{b_1,\ldots,b_n\}$ a linearly independent subset of ${}^{\circ}A$, by 4.2, $a_1 = \ldots = a_n = 0$. Thus $u = 0$. □

Therefore, a C*-algebra A is prime if and only if $A \otimes {}^{\circ}A \cong El(A)$ via θ.

We now answer the question of the uniqueness of a representation of an elementary operator by a sum of multiplications. To this end, we introduce the following notions.

Definition 4.6. Let S be an elementary operator on the algebra A. If $S = \sum_{j=1}^{n} M_{a_j,b_j}$ for some elements a_j, b_j in A then S is said to have *a representation of length* n. The representation $S = 0$ is defined to have length zero. The minimum of all lengths of representations of S is called *the minimal length of* S and denoted by $\ell(S)$. Any representa-

tion of length $\ell(S)$ is called *a minimal representation of* S.

A characterization of minimal representations is derived from Theorem 4.2 as follows.

Proposition 4.7. *Let* A *be an ultraprime normed algebra and let* $S \in E\ell(A)$. *The representation* $S = \sum_{j=1}^{n} M_{a_j,b_j}$ *is minimal if and only if the sets* $\{a_1,\ldots,a_n\}$ *and* $\{b_1,\ldots,b_n\}$ *are linearly independent.*

Proof. Clearly, if the representation $S = \sum_{j=1}^{n} M_{a_j,b_j}$ is minimal then both $\{a_1,\ldots,a_n\}$ and $\{b_1,\ldots,b_n\}$ have to be linearly independent which does not use the primeness of A. Conversely, suppose that $\{a_1,\ldots,a_n\}$ and $\{b_1,\ldots,b_n\}$ are linearly independent and let $S = \sum_{k=1}^{m} M_{c_k,d_k}$ be a minimal representation of S . Assuming that $\{b_1,\ldots,b_n,d_1,\ldots,d_r\}$ with $r \geq 1$ is a maximal linearly independent subset of $\{b_1,\ldots,b_n,d_1,\ldots,d_m\}$ we deduce from

$$\sum_{j=1}^{n} M_{a_j,b_j} + \sum_{k=1}^{m} M_{-c_k,d_k} = 0$$

and Theorem 4.2 that $r < m$ and

$$(4.7.1) \qquad a_j = -\sum_{k=r+1}^{m} \lambda_{kj}(-c_k) \quad \text{for all} \quad 1 \leq j \leq n$$

and

$$(4.7.2) \qquad -c_j = -\sum_{k=r+1}^{m} \lambda_{kj}(-c_k) \quad \text{for all} \quad 1 \leq j \leq r$$

where λ_{kj} , $1 \leq j \leq n+r$, $r+1 \leq k \leq m$, are complex numbers such that $d_k = \sum_{j=1}^{n} \lambda_{kj} b_j + \sum_{j=1}^{r} \lambda_{k(j+n)} d_j$ for all $r+1 \leq k \leq m$. However, equation (4.7.2) contradicts the linear independency

of $\{c_1,\ldots,c_m\}$. Thus, $\{b_1,\ldots,b_n\}$ is maximal linearly independent in $\{b_1,\ldots,b_n,d_1,\ldots,d_m\}$, and (4.7.1) holds with $r = 0$. Therefore, $n = \dim \operatorname{lin}\{a_1,\ldots,a_n\} \leq \dim \operatorname{lin}\{c_1,\ldots,c_m\} = m$ and the minimality of m yields $n = m = \ell(S)$.□

As a consequence, minimal representations are unique in the following sense.

Corollary 4.8. *Let* A *be an ultraprime normed algebra. If* $S = \sum_{j=1}^{n} M_{a_j,b_j}$ *and* $S = \sum_{k=1}^{n} M_{c_k,d_k}$ *are minimal representations of* $S \in L\ell(A)$, *then there exists a unique invertible complex* $n\times n$*-matrix* $(\lambda_{kj})_{1\leq j,k\leq n}$ *such that*

$$a_j = \sum_{k=1}^{n} \lambda_{kj} c_k \quad \text{and} \quad d_k = \sum_{j=1}^{n} \lambda_{kj} b_j \quad \text{for all} \quad 1 \leq j,k \leq n.$$

Proof. The existence of the matrix $\Lambda = (\lambda_{kj})_{1\leq j,k\leq n}$ follows from the proof of 4.7, and the uniqueness and invertibility of Λ are consequences of the linear independency of the sets $\{a_j\}$, etc.□

Hence, with respect to some fixed n-tuples $a,b \in A^n$, each minimal representation of $S = \sum_{j=1}^{n} M_{a_j,b_j}$ with $n = \ell(S)$ is in one-to-one correspondence with an invertible $n\times n$-matrix over $\mathbb{C}$. We shall use this correspondence in the next chapter to investigate positivity properties of elementary operators on prime C*-algebras.

If $S = \sum_{j=1}^{n} M_{a_j,b_j}$ is an arbitrary representation of the elementary operator $S \neq 0$ on the ultraprime algebra A, by carrying out the first step in the proof of Theorem 4.2 twice, we obtain linearly independent sets $\{a_1',..,a_r'\} \subseteq \operatorname{lin}\{a_1,..,a_n\}$ and $\{b_1',\ldots,b_r'\} \subseteq \operatorname{lin}\{b_1,\ldots,b_n\}$ for some $1 \leq r \leq n$ such that $S = \sum_{j=1}^{r} M_{a_j',b_j'}$. By Proposition 4.7, this representation now

is minimal. By means of this we can also compare two different not necessarily minimal representations of the same elementary operator S, so it suffices to deal with minimal representations.

The remainder of this chapter is devoted to some more applications of Theorem 4.2. Observe at first that, on a prime algebra A, the multiplication $M_{a,b}$ is nilpotent if and only if a is nilpotent or b is nilpotent. If A is ultraprime, the same statement holds with 'quasinilpotent' in place of 'nilpotent'. Indeed, $c_M \|a^n\| \|b^n\| \leq \|M_{a,b}^n\| \leq \|a^n\| \|b^n\|$ for some $0 < c_M \leq 1$ by Lemma 1.1 from which $r(a)\, r(b) = r(M_{a,b})$ follows. (Here and in the sequel, $r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$ shall denote the spectral radius of an element x in a normed algebra.) The equation $r(a)\, r(b) = r(M_{a,b})$ is well known in the case $A = L(E)$ (see e.g. [15], thm 0.7.1). Concerning algebraic elements we have the following.

Proposition 4.9. *Let A be a unital ultraprime normed algebra. If $a,b \in A$ and b is not algebraic, then the following conditions are equivalent:*

(a) $M_{a,b}$ *is algebraic.*

(b) $M_{a,b}$ *is nilpotent.*

(c) a *is nilpotent.*

Proof. Evidently (c) => (b) => (a). If $p(\zeta) = \sum_{j=0}^{n} \alpha_j \zeta^j$ is a complex polynomial with $\alpha_n = 1$ and $p(M_{a,b}) = \sum_{j=0}^{n} M_{\alpha_j a^j, b^j} = 0$ then, by 4.2, $\alpha_0 = \alpha_1 a = \ldots = \alpha_{n-1} a^{n-1} = a^n = 0$ since $\{1,b,\ldots,b^n\}$ is linearly independent. In particular, a is nilpotent. Thus (a) => (c). □

In a series of papers (see [56] for a survey) Fialkow studied properties of *generalized inner derivations* $\delta_{a,b}$ given by $\delta_{a,b}x = ax - xb$. As another consequence of Theorem 4.2 we will answer the question under which circumstances the product of two generalized inner derivations is again a generalized inner derivation. The analogous question for (inner) derivations will be discussed in the Supplement to this chapter.

Proposition 4.10. *Let* A *be a unital ultraprime normed algebra, and let* $a_1, a_2, b_1, b_2 \in A$. *Then* $\delta_{a_1,b_1}\delta_{a_2,b_2} = \delta_{c,d}$ *for some* $c,d \in A$ *if, and only if, one of the following three cases occurs:*

(i) *Both* δ_{a_1,b_1} *and* δ_{a_2,b_2} *are left multiplications.*

(ii) *Both* δ_{a_1,b_1} *and* δ_{a_2,b_2} *are right multiplications.*

(iii) $\{\delta_{a_1,-b_1}, \delta_{a_2,b_2}, id_A\}$ *is linearly dependent.*

In view of 4.2 the proof reduces to a tedious repetition of the arguments for the case $A = L(E)$ in [64], prop. 1, and is hence omitted.

We now turn our attention to hermitian operators on C*-algebras. Recall that an element a in a complex unital normed algebra A is called *hermitian* if $\|e^{ita}\| = 1$ for each $t \in \mathbb{R}$, or, equivalently, if its *numerical range* $V(a) = \{f(a) \mid f \in A^*, \|f\| = f(1) = 1\}$ is contained in $\mathbb{R}$. (Here, A^* denotes the topological dual of A.) Since the left and the right regular representation of A are both isometries, $a \in A$ is hermitian if and only if L_a and R_a are hermitian operators on A. If A is a unital C*-algebra, an easy argument shows that $a \in A$ is hermitian if and only if $a = a^*$ (cf. e.g. [20], 12.20). For any unital Banach algebra A we will denote by A_h the space of all hermitian elements in A and

by $S(A) = \{f \in A^* \mid \|f\| = f(1) = 1\}$ the set of all *states* on A.

Our next aim is to show that on certain prime C*-algebras the square of a hermitian operator T is rarely hermitian. We start with the following lemma.

Lemma 4.11. *Let* A *be a unital C*-algebra, and let* $a,b \in A$. *The generalized inner derivation* $\delta_{a,b}$ *is hermitian if and only if there are* $a',b' \in A_h$ *such that* $\delta_{a,b} = \delta_{a',b'}$.

Proof. If $a,b \in A_h$ then $\exp(it\delta_{a,b}) = \exp(itL_a)\exp(-itR_b)$ is an isometry for every real t, i.e. $\delta_{a,b}$ is hermitian. Suppose now $\delta_{a,b}$ is hermitian. If $\phi \in S(A)$ then $f:L(A) \to \mathbb{C}$, $f(T) = \phi(T(1))$ is a state on $L(A)$. Therefore $\phi(a - b) = \phi(\delta_{a,b}1) = f(\delta_{a,b}) \in \mathbb{R}$. Hence, $V(a - b) \subseteq \mathbb{R}$, i.e. $a - b$ is hermitian. It follows by [20], 10.4 that $\delta_{a,b} - L_{a-b}$ is hermitian; moreover $\delta_{a,b} - L_{a-b} = \delta_b$ is the inner derivation $\delta_b x = bx - xb$. Since $\delta_b 1 = 0$, $\exp(it\delta_b)$ is an isometry on A with $\exp(it\delta_b)1 = 1$. By [91], lem. 8, $\exp(it\delta_b)(x^*) = (\exp(it\delta_b)x)^*$ for all $x \in A$. Therefore

$$i\delta_b x^* = \lim_{t\to\infty} \frac{1}{t}(\exp(it\delta_b)(x^*) - x^*)$$
$$= \lim_{t\to\infty} \frac{1}{t}(\exp(it\delta_b)x - x)^* = -i(\delta_b x)^*,$$

or, equivalently, $\delta_b x^* = -(\delta_b x)^*$ for all $x \in A$. Thus, for each $x \in A_h$, $bx - xb = -(bx - xb)^* = b^*x - xb^*$, i.e. $(b - b^*)x = x(b - b^*)$ which shows that $b - b^*$ is central. Hence $\operatorname{Im} b = \frac{b - b^*}{2i}$ is central so that $\delta_b = \delta_{b'}$ with $b' = b - i\operatorname{Im} b = \operatorname{Re} b \in A_h$.

We now put $a' = a - b + b' \in A_h$ and obtain

$$\delta_{a',b'} = L_{a-b} + L_{b'} - R_{b'} = L_{a-b} + \delta_{b'} = \delta_{a,b}.$$

This accomplishes the proof. □

The next result is essentially contained in [123], remark 3.5. It can be considered as a generalization of Sakai's theorem stating that every derivation on a W*-algebra is inner ([121], 4.1.6).

Proposition 4.12. *Let* A *be a W*-algebra. Then every hermitian operator on* A *is a generalized inner derivation* $\delta_{a,b}$ *with* $a,b \in A_h$.

Proof. Let $T \in L(A)_h$. As in the proof of 4.11, $\delta = T - L_{T1}$ is a hermitian operator on A, and so $e^{it\delta}$ is an isometry with $e^{it\delta}1 = 1$ for each real t. By [91], thm 7, $e^{it\delta}$ is a Jordan *-isomorphism for all t, hence δ is a Jordan *-derivation on A. By [123], thm 3.3, δ is actually a *-derivation. We now apply Sakai's theorem to obtain $b \in A_h$ such that $\delta = \delta_b$. Put $a = T1 + b \in A_h$. Then

$$T = L_{T1} + \delta = L_{a-b} + L_b - R_b = L_a - R_b = \delta_{a,b}$$

which completes the proof. □

Corollary 4.13. *Let* A *be a C*-algebra. Then every hermitian operator on* A *is of the form* $\delta_{a,b}$ *with* $a,b \in A_h^{**}$.

Proof. If $T \in L(A)$ is hermitian, its second adjoint $T'' \in L(A^{**})$ is hermitian too. Since the second dual A^{**} of A can be identified with the universal enveloping W*-algebra of A, the result follows by 4.12. □

Clearly, Proposition 4.12 holds in any unital C*-algebra which has no outer derivations, e.g. in any simple unital C*-algebra ([121], 4.1.11). This is used in our next result which extends the case $A = L(H)$ considered in [64], prop. 5. Since 4.12 is only applied to prime C*-algebras where every Jordan derivation is automatically a derivation ([81], thm 3.3) we note that

Proposition 4.14 is in fact independent of Sinclair's theorem ([123], thm 3.3).

Proposition 4.14. *Let* A *be a prime unital* C^**-algebra without outer derivations. If* $T \in L(A)$ *is hermitian and* T^2 *is also hermitian, then* $T = L_a$ *or* $T = R_b$ *for some* $a,b \in A_h$.

Proof. By the proof of 4.12 respectively the above remarks, both $T - L_{T1}$ and $T^2 - L_{T^21}$ are *-derivations on A. By hypothesis, there are $a,c \in A_h$ such that $T - L_{T1} = \delta_a$ and $T^2 - L_{T^21} = \delta_c$. Therefore, $T = \delta_a + L_{T1} = L_a - R_a + L_{T1} = L_{a+T1} - R_a = \delta_{b,a}$ and $T^2 = \delta_c + L_{T^21} = L_c - R_c + L_{T^21} = L_{c+T^21} - R_c = \delta_{d,c}$ where both $b = a + T1$ and $d = c + T^21$ belong to A_h since $T1, T^21 \in A_h$. Hereby, $\delta_{b,a}^2 = L_{b^2} - 2M_{b,a} + R_{a^2} = \delta_{d,c}$, i.e.

$$M_{b^2-d,1} + M_{-2b,a} + M_{1,a^2+c} = 0.$$

It now follows from Theorem 4.2 that the set $\{1,a,a^2 + c\}$ must be linearly dependent. If $a \in \mathbb{C}1$ then $T = L_{T1}$ is the left multiplication by $T1 \in A_h$. If $a \notin \mathbb{C}1$ then $\{1,a\}$ is maximal linearly independent in $\{1,a,a^2 + c\}$ and $a^2 + c = \lambda_1 + \lambda_2 a$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. Applying 4.2 again, we obtain $b^2 - d = -\lambda_1 1$ and $2b = \lambda_2 \cdot 1$. Thus, $T = L_b - R_a = L_{\lambda_2/2} - R_a = R_{a-\lambda_2/2}$ the right multiplication by $a - \lambda_2/2 \in A_h$. □

The following result was observed in [5], ex. 5.9 for $a = b$ a non-trivial projection in $A = L(H)$.

Corollary 4.15. *Let* A *be a prime unital* C^**-algebra without outer derivations. If* $a,b \in A_h$ *are non-scalar, then* $\delta_{a,b}$ *is hermitian while* $\delta_{a,b}^2$ *is not hermitian.*

Proof. If both $\delta_{a,b}$ and $\delta_{a,b}^2$ were hermitian, by 4.14, there is $c \in A_h$ such that, say $\delta_{a,b} = L_c$. Therefore, $\delta_{a-c,b}$

$= 0$ which implies $a - c = b$ and hence $L_{a-c} = R_b = L_b$. Thus $b \in Z(A) = \mathbb{C}1$, a contradiction. □

We conclude the discussion of hermitian operators with a characterization of hermitian multiplications. Again, this was proved by Anderson and Foiaş ([5], thm 5.8) in the case $a = b$ a projection in $A = L(H)$.

Corollary 4.16. *Let* A *be a prime unital* C^**-algebra without outer derivations, and let* $a,b \in A$. *Then* $M_{a,b}$ *is hermitian if and only if* $M_{a,b}$ *is a left multiplication or a right multiplication by a self-adjoint element in* A.

Proof. If $M_{a,b}$ is hermitian then, following the proofs of 4.11 and 4.12, $ab \in A_h$ and $M_{a,b} - L_{ab} = \delta_c$ for some $c \in A_h$. Thus

$$M_{a,b} + M_{ab+c,-1} + M_{1,c} = 0$$

and by Theorem 4.2 the set $\{-1,b,c\}$ must be linearly dependent. If $b \in \mathbb{C}1$ then $M_{a,b} = L_{ab}$ with $ab \in A_h$. Otherwise, $c = \lambda_1 + \lambda_2 b$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$ and $a = \lambda_1 \cdot 1$, $ab + c = -\lambda_2 \cdot 1$. Therefore, $M_{a,b} = R_{ab}$ with $ab \in A_h$. □

Supplement 4.17. A result on derivations on prime rings.

In [64] Fong and Sourour also treated the question of when the product of two derivations on $L(E)$ will again be a derivation, and they used their version of Theorem 4.2 to derive an answer (ibid., prop. 2). Unfortunately, their proof has a gap since it uses a wrongly stated lemma (Lemma 2 of [64]). As we were informed by A.R.Sourour (private communication) this gap can be closed. But it should be emphazised that this result is not really a consequence of the result on elementary operators. In fact, a number of years earlier Posner proved a much more general theorem which we recall here since it seems to have

fallen into oblivion somehow (see also [146]).

Theorem ([117], thm 1). *Let* δ_1, δ_2 *be two derivations on a prime ring* R *of characteristic different from* 2. *If* $\delta_1\delta_2$ *is also a derivation then* $\delta_1 = 0$ *or* $\delta_2 = 0$.

Proof. By hypothesis, we have for all $x,y \in R$

$$\delta_1\delta_2(xy) = ((\delta_1\delta_2)x)y + x(\delta_1\delta_2 y).$$

On the other hand,

$$\begin{aligned}\delta_1\delta_2(xy) &= \delta_1(\delta_2(xy)) \\ &= \delta_1((\delta_2 x)y + x\,\delta_2 y) \\ &= (\delta_1(\delta_2 x))y + (\delta_2 x)(\delta_1 y) + (\delta_1 x)(\delta_2 y) + x(\delta_1\delta_2 y).\end{aligned}$$

Hence, for all $x,y \in R$,

$$\delta_2 x\ \delta_1 y + \delta_1 x\ \delta_2 y = 0. \tag{1}$$

Replacing x by xz we obtain

$$\delta_2(xz)\ \delta_1 y + \delta_1(xz)\ \delta_2 y = 0$$

and thus

$$(\delta_2 x)\ z\ (\delta_1 y) + x\ \delta_2 z\ \delta_1 y + (\delta_1 x)\ z\ (\delta_2 y) + x\ \delta_1 z\ \delta_2 y = 0.$$

Hence, for all $x,y,z \in R$,

$$(\delta_2 x)\ z\ (\delta_1 y) + (\delta_1 x)\ z\ (\delta_2 y) = 0.$$

Replacing z by $\delta_1 w$ this yields

$$\delta_2 x\ \delta_1 w\ \delta_1 y + \delta_1 x\ \delta_1 w\ \delta_2 y = 0.$$

Using (1), we obtain

$$-\delta_1 x\ \delta_2 w\ \delta_1 y + \delta_1 x\ \delta_1 w\ \delta_2 y = 0,$$

thus

$$\delta_1 x(-\delta_2 w\ \delta_1 y + \delta_1 w\ \delta_2 y) = 0.$$

Using (1) again, we conclude

$$2\ \delta_1 x\ \delta_2 w\ \delta_1 y = 0$$

and thus

(2) $$\delta_1 x\ \delta_2 w\ \delta_1 y = 0$$

for all $x,y,w \in R$.

Replacing x by xz and using (2), we obtain for all $x,y,w,z \in R$

$$(\delta_1 x)\ z\ \delta_2 w\ \delta_1 y = 0.$$

Since R is prime, this implies $\delta_1 x = 0$ for all $x \in R$ or $\delta_2 w\ \delta_1 y = 0$ for all $w,y \in R$. Suppose the second case. Replacing w by wz, we obtain

$$(\delta_2 w)\ z\ (\delta_1 y) = 0$$

for all $y,w,z \in R$

forcing $\delta_2 w = 0$ for all $w \in R$ or $\delta_1 y = 0$ for all $y \in R$. This proves the result. □

Remarks. 1. The assumption can be formally weakened to δ_1, δ_2 or $\delta_1 \delta_2$ a Jordan derivation, since every Jordan derivation on a prime ring with characteristic different from 2 is a derivation ([81], thm 3.3).

2. If $\delta_1 = \delta_2$ it suffices to assume R to be semiprime (see [82], proof of lem. 1.1.9).

3. In case of characteristic 2, the result is false. E.g. take $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, a 2×2 matrix over a field of characteristic 2. Then $a^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\delta_a{}^2 x = \delta_a(ax - xa) = a^2 x + xa^2 - 2axa = 0$ for all x.

But $\delta_a \neq 0$, for example $\delta_a \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Corollary. *If* δ *is a non-zero derivation on a prime ring* R *of characteristic different from* 2, *then the commutant* $(\delta R)'$ *of* δR *is trivial.*

Proof. Take $b \in (\delta R)'$. Then $\delta_b \delta = 0$ which implies $\delta_b = 0$ by the Theorem. Hence, $b \in Z(R)$.□

This corollary is due to J.P.Williams ([146], cor. 1 on p. 275) in the case $R = L(H)$. Indeed, it is his original argument presented here. It follows at once that the ranges of any two non-zero inner derivations intersect non-trivially.

5. POSITIVITY OF ELEMENTARY OPERATORS

The present chapter is devoted to a weakening of the equality $S = 0$ studied in the previous chapter to an inequality $S \geq 0$, i.e. to an investigation of positivity properties of elementary operators on partially ordered Banach algebras. We will concentrate here on C*-algebras. As already indicated, the most important tool will be Theorem 4.2.

Let us begin with some remarks on the order structure of a C*-algebra A. It has turned out that, for several reasons (see e.g. [52], [100], [101], [96]), merely studying positive operators on A (those leaving the positive cone A_+ of A invariant) is not adequate. For each $n \in \mathbb{N}$ one associates to every $T \in L(A)$ an operator $T^{(n)} \in L(M_n(A))$ given by $T^{(n)}(a_{ij})_{1\leq i,j\leq n} = (Ta_{ij})_{1\leq i,j\leq n}$ for any $(a_{ij})_{1\leq i,j\leq n}$ in $M_n(A)$, the C*-algebra of all $n\times n$-matrices over A (cf. [136], IV.3), and calls T *n-positive* if $T^{(n)}$ is positive. Obviously, 1-positivity is the same as positivity. The operator T is called *completely positive* if T is n-positive for every $n \in \mathbb{N}$. Since their inception by Stinespring in 1955 [128], complete positivity and n-positivity have been studied extensively, and one of the fundamental results is the following.

Theorem (Stinespring[128], Størmer[129], Choi[28]).

For every C-algebra* A *the following are equivalent:*

(a) *Every positive operator on* A *is 2-positive.*

(b) *Every positive operator on* A *is completely positive.*

(c) A *is commutative.*

Thus, on a non-commutative C*-algebra there are always many positive operators which are not n-positive for some $n \geq 2$. Another important result is Stinespring's representation theorem ([128], thm 1; see also [97]) for completely positive operators.

All these results have been extended and generalized to linear mappings on the duals of C*-algebras and on the preduals of W*-algebras, and many other contributions were made (see e.g. [29], [32], [84], [133], [134], [135], [139], [140], see also the references in Choi's survey article [30]). Very recently non-linear completely positive mappings have been studied, too ([6]).

In [29], thm 1, Choi proved that a linear mapping T on the algebra M_n of all complex $n \times n$-matrices is completely positive if and only if T is an elementary operator of the form $\sum_j M_{c_j^*, c_j}$ where c_j are finitely many suitably choosen $n \times n$-matrices. Some years before, Kraus had shown ([96], thm 3.3) that any normal completely positive contraction on $L(H)$ can be approximated by completely positive elementary operators $\sum_{j=1}^{n} M_{c_j^*, c_j}$ with $c_j \in L(H)$ in the pointwise strong operator topology.

In [110], thm 2.4, the positivity of multiplications on C*-algebras was characterized as follows.

5.1. *If $M_{a,b}$ is a positive multiplication on a C*-algebra A then there exists $c \in A$ such that $M_{a,b} = M_{c^*,c}$. Hence $M_{a,b}$ is completely positive.*

For prime C*-algebras this is an immediate consequence of Theorem 4.2:

If $M_{a,b}x \geq 0$ for all $x \in A_+$ then $M_{a,b} - M_{b^*,a^*} = 0$. By 4.2, the set $\{a,b^*\}$ is linearly dependent, hence there is $\lambda \in \mathbb{C}$ such that $a = \lambda b^*$ unless $b = 0$ in which case the assertion is trivial. Since $0 \leq ab = \lambda b^*b$, we have $\lambda \geq 0$ and may choose $c = \sqrt{\lambda}\, b$.

Thus, by denoting the positive respectively completely positive operators in a subset $X \subseteq L(A)$ by X_p respectively X_{cp} and using Corollary 4.4, the results of Choi and Kraus become:

$$L(A)_{cp} = E\ell_{cp}(A) = \text{conv } M(A)_p \quad \text{for } A = M_n$$

and

$$\{T \in L(A)_{cp} \mid \|T\| \leq 1,\ T \text{ normal}\} = \overline{\text{conv}}\ M(A)_p \quad \text{for } A = L(H)$$

where conv stands for *convex hull* and the closure is to be taken in the pointwise strong operator topology.

The main results of this chapter consist in an extension of the equality $E\ell_{cp}(A) = \text{conv } M(A)_p$ to a wider class of C*-algebras and a generalization of 5.1 to arbitrary elementary operators. The latter, however, needs some stronger positivity assumptions depending on the length of an elementary operator. We make no attempt to characterize arbitrary *positive* elementary operators since the identity

$$L(A)_p = E\ell_p(A) = (\text{conv } M(A))_p$$

holding for $A = M_n$ (Cor. 4.4) shows that such a characterization is hardly achievable.

Our first aim is a description of the hermitian-preserving elementary operators on a prime C*-algebra. For $A = M_n$ this was obtained by Hill([83], see also [45] and [84]).

Proposition 5.2. *The following conditions on an elementary operator* S *on a prime C*-algebra* A *are equivalent:*

(a) $SA_h \subseteq A_h$

(b) $S = \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_{kj} M_{b_k^*, b_j}$ *for some elements* $b_1, \ldots, b_n \in A$ *and a hermitian* $n \times n$*-matrix* $(\lambda_{kj})_{1 \leq j,k \leq n}$.

Proof. Obviously (b) implies (a). To prove the converse we take a minimal representation $S = \sum_{j=1}^{n} M_{a_j, b_j}$ of S. By hypothesis we have

$$\sum_{j=1}^{n} M_{a_j, b_j} = \sum_{k=1}^{n} M_{b_k^*, a_k^*}.$$

Thus, by 4.8, there exists $\Lambda = (\lambda_{kj})_{1 \leq j,k \leq n} \in M_n$ such that

$$a_j = \sum_{k=1}^{n} \lambda_{kj} b_k^* \quad \text{and} \quad a_k^* = \sum_{j=1}^{n} \lambda_{kj} b_j$$

for all j and k. Therefore,

$$a_j = \sum_{k=1}^{n} \lambda_{kj} b_k^* = \sum_{k=1}^{n} \overline{\lambda_{jk}} b_k^*$$

and the linear independency of $\{b_1, \ldots, b_n\}$ yields $\Lambda = \Lambda^*$. Also,

$$S = \sum_{j=1}^{n} M_{a_j, b_j} = \sum_{j=1}^{n} M_{\sum_{k=1}^{n} \lambda_{kj} b_k^*, b_j} = \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_{kj} M_{b_k^*, b_j}. \square$$

It follows that a hermitian-preserving elementary operator is 'diagonalizable'.

Corollary 5.3. *If* S *is a hermitian-preserving elementary operator of minimal length* $n \geq 1$ *on a prime C*-algebra* A *then there exist a linearly independent subset* $\{b_1, \ldots, b_n\}$ *of* A *and non-zero real numbers* $\lambda_1, \ldots, \lambda_n$ *with the property that*

$$S = \sum_{j=1}^{n} \lambda_j M_{b_j^*, b_j}.$$

Proof. By 5.2 and 4.8 there exist an invertible hermitian $n\times n$-matrix $\Lambda = (\lambda_{k\ell})_{1\le k,\ell\le n}$ and a linearly independent subset $\{\tilde{b}_1,\dots,\tilde{b}_n\}$ of A such that $S = \sum_{k,\ell=1}^{n} \lambda_{k\ell} M_{\tilde{b}_k^*,\tilde{b}_\ell}$. Take a unitary $n\times n$-matrix $U = (\mu_{ij})_{1\le i,j\le n}$ such that $U^*\Lambda U$ is diagonal and let $\lambda_j = \sum_{k,\ell=1}^{n} \lambda_{k\ell}\overline{\mu_{kj}}\mu_{\ell j}$ denote the eigenvalues of Λ. Defining $b_j = \sum_{\ell=1}^{n} \overline{\mu_{j\ell}}\tilde{b}_\ell$ we obtain a linearly independent subset $\{b_1,\dots,b_n\}$ of A such that $\tilde{b}_\ell = \sum_{j=1}^{n} \mu_{\ell j} b_j$ for all $1 \le \ell \le n$. Then,

$$
\begin{aligned}
S &= \sum_{k,\ell=1}^{n} \lambda_{k\ell} M_{\tilde{b}_k^*,\tilde{b}_\ell} \\
&= \sum_{k,\ell=1}^{n} \lambda_{k\ell} M_{\sum_{i=1}^{n}\overline{\mu_{ki}}b_i^*,\ \sum_{j=1}^{n}\mu_{\ell j}b_j} \\
&= \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{n} \lambda_{k\ell}\overline{\mu_{ki}}\mu_{\ell j} M_{b_i^*,b_j} \\
&= \sum_{j=1}^{n} \lambda_j M_{b_j^*,b_j} \ . \ \square
\end{aligned}
$$

Before we go on to characterize completely positive elementary operators we exemplify the above method by the transpose map on the 2×2-matrices.

Example 5.4. Let $A = M_2$. The transpose map τ is a well known example of a positive but not 2-positive linear mapping on A. As an elementary operator, τ has a representation

$$\tau = \sum_{i,j=1}^{2} M_{u_{ij},u_{ij}} \tag{5.4.1}$$

where $u_{ij} = (\delta_{ik}\delta_{\ell j})_{1\le k,\ell\le 2}$ for $i,j = 1,2$ denotes the usual matrix unit of A. Since $\{u_{ij} \mid 1 \le i,j \le 2\}$ is linearly independent, by Prop. 4.7, $\ell(\tau) = 4$. To obtain a representation

(5.4.2) $$\tau = \sum_{j,k=1}^{4} \lambda_{jk} M_{\tilde{b}_k^*, \tilde{b}_j}$$

as asserted in Proposition 5.2 we have to solve the equation

$$\begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{pmatrix} = \Lambda \begin{pmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{pmatrix}$$

and find

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, the representation (5.4.1) can also serve as a representation of the form (5.4.2) (with $\tilde{b}_1 = u_{11}$, $\tilde{b}_2 = u_{21}$, $\tilde{b}_3 = u_{12}$, $\tilde{b}_4 = u_{22}$). Now, Λ is diagonalized to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{via the unitary} \quad U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \nu & \nu & 0 \\ 0 & \nu & -\nu & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\nu = \frac{1}{\sqrt{2}}$. By Corollary 5.3, $b_j = \sum_{\ell=1}^{4} \overline{u_{j\ell}}\, \tilde{b}_\ell$ for $1 \le j \le 4$, hence $b_1 = u_{11}$, $b_2 = \frac{1}{\sqrt{2}}(u_{21} + u_{12})$, $b_3 = \frac{1}{\sqrt{2}}(u_{12} - u_{21})$ and $b_4 = u_{22}$. Therefore, the final representation is

(5.4.3) $$\tau = M_{u_{11},u_{11}} + \tfrac{1}{2} M_{u_{12}+u_{21},u_{21}+u_{12}} - \tfrac{1}{2} M_{u_{21}-u_{12},u_{12}-u_{21}} + M_{u_{22},u_{22}}.$$

This example disproves the naïve conjecture that an elementary operator S is positive if and only if the associated matrix Λ is positive. In fact, Choi proved in [29] (see also [84]) that Λ is positive if and only if S is completely positive. We will now extend this result from the $n \times n$-matrices to prime C*-algebras. This as well as Theorem 5.7 was obtained together with B.Kümmerer (see [98]).

Theorem 5.5. *Let* S *be an elementary operator of minimal length* $n \geq 1$ *on a prime C*-algebra* A. *Then* S *is completely positive if and only if there is a linearly independent subset* $\{c_1,\dots,c_n\}$ *in* A *such that* $S = \sum_{j=1}^{n} M_{c_j^*,c_j}$.

Proof. The "if"-part is clear. To prove the "only if"- part suppose at first that A acts irreducibly on the Hilbert space H and that S is represented as $S = \sum_{j=1}^{n} \lambda_j M_{b_j^*,b_j}$ with non-zero real numbers λ_j and $\{b_1,\dots,b_n\}$ linearly independent (Corollary 5.3). Let K denote the orthogonal sum of n copies of H, v the bounded operator from H into K defined by $v\xi = (b_1\xi,\dots,b_n\xi)$ with adjoint $v^* \in L(K,H)$ given by $v^*(\xi_1,\dots,\xi_n) = \sum_{j=1}^{n} b_j^*\xi_j$, and let $\Lambda = 1 \otimes \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ belong to $L(H) \otimes M_n = L(K)$. Then A acts in a natural way on K and for $\xi = \sum_{k=1}^{m} x_k v\xi_k \in AvH$ we have

$$
\begin{aligned}
(\Lambda\xi \mid \xi) &= (\Lambda \sum_{k=1}^{m} x_k v\xi_k \mid \sum_{\ell=1}^{m} x_\ell v\xi_\ell) \\
&= \sum_{k,\ell=1}^{m} (\Lambda x_k v\xi_k \mid x_\ell v\xi_\ell) \\
&= \sum_{k,\ell=1}^{m} (v^* x_\ell^* x_k \Lambda v\xi_k \mid \xi_\ell) \\
&= \sum_{k,\ell=1}^{m} (v^* x_\ell^* x_k \Lambda (b_1\xi_k,\dots,b_n\xi_k) \mid \xi_\ell) \\
&= \sum_{k,\ell=1}^{m} (v^* x_\ell^* x_k (\lambda_1 b_1\xi_k,\dots,\lambda_n b_r\xi_k) \mid \xi_\ell) \\
&= \sum_{k,\ell=1}^{m} (v^* (\lambda_1 x_\ell^* x_k b_1\xi_k,\dots,\lambda_n x_\ell^* x_k b_n\xi_k) \mid \xi_\ell) \\
&= \sum_{k,\ell=1}^{m} (\sum_{j=1}^{n} \lambda_j b_j^* x_\ell^* x_k b_j \xi_k \mid \xi_\ell)
\end{aligned}
$$

$$= \sum_{k,\ell=1}^{m} (S(x_\ell^* x_k)\xi_k | \xi_\ell) \geq 0$$

by the complete positivity of S.

Since $\{b_1,\ldots,b_n\}$ is linearly independent, the space vH is separating for the commutant $A' = \mathbb{C}1 \otimes M_n$ of A in $\mathcal{L}(K)$, thus vH is cyclic, i.e. $\overline{AvH} = K$. Hence, Λ is positive on a dense subset of K and so $\lambda_j > 0$ for all j. We can now choose $c_j = \sqrt{\lambda_j}\, b_j$.

If A is an arbitrary prime C*-algebra and $S = \sum_{j=1}^{n} \lambda_j M_{b_j^*, b_j}$ let B_o denote the separable C*-subalgebra of A generated by $\{b_1,\ldots,b_n\}$. By [50], prop. 3.1, there is a separable prime C*-subalgebra B containing B_o. Since $SB \subseteq B$ and B is primitive by Dixmier's theorem ([116], 4.3.6), we can apply the first part of the proof to complete the argument. □

From this theorem we derive a characterization of the extreme points of convex sets of completely positive elementary operators. For any C*-algebra A and $c \in A_+$ denote by $E\ell_{cp}(A;c)$ the convex set of all completely positive elementary operators S on A satisfying $S1 = c$; this is clearly unambiguous for non-unital algebras, too.

Corollary 5.6. *Let A be a prime C*-algebra and let $c \in A_+\backslash\{0\}$. Then $S \in E\ell_{cp}(A;c)$ is an extreme point if and only if there exist $c_1,\ldots,c_n \in A$ such that $S = \sum_{j=1}^{n} M_{c_j^*, c_j}$ and the set $\{c_i^* c_j \mid 1 \leq i, j \leq n\}$ is linearly independent. In particular, every positive multiplication M with $M1 = c$ is an extreme point in $E\ell_{cp}(A;c)$.*

Proof. We adopt the notation used in 5.5. If $S \in E\ell_{cp}(A;c)$ with $\ell(S) = n$ then, by 5.5, there is a separable primitive

C*-subalgebra B of A and a linearly independent subset $\{c_1,\dots,c_n\}$ in B such that $S = \sum_{j=1}^{n} M_{c_j^*,c_j}$. Clearly, if S is an extreme point in $E\ell_{cp}(A;c)$ then S is also an extreme point in $E\ell_{cp}(B;c)$.

On the other hand, if S is not an extreme point in $El_{cp}(A;c)$, say $S = \frac{1}{2}(S_1 + S_2)$ for some elementary operators

$$S_1 = \sum_{k=1}^{r} M_{a_k^*,a_k},\quad S_2 = \sum_{\ell=1}^{s} M_{b_\ell^*,b_\ell} \in E\ell_{cp}(A;c)$$

we may consider the C*-subalgebra generated by $\{a_k \mid 1 \le k \le r\}$, $\{b_\ell \mid 1 \le \ell \le s\}$ and $\{c_j \mid 1 \le j \le n\}$ and enlarge it to a primitive C*-subalgebra B_1 of A. Since S, S_1 and S_2 leave B_1 invariant, S cannot be an extreme point in $E\ell_{cp}(B_1;c)$.

Replacing A by either B or B_1 we can therefore assume that A acts irreducibly on some Hilbert space H.

Since A', the commutant of A in $L(K)$, is isomorphic to M_n, for every $y \in A'$ there is a unique $n\times n$-matrix $(\gamma_{ij})_{1\le i,j\le n}$ such that $\sum_{i,j=1}^{n} \gamma_{ij} c_i^* c_j = w^* y w$ where $w \in L(H,K)$ is given by $w\xi = (c_1\xi,\dots,c_n\xi)$. Therefore, $\{c_i^*c_j \mid 1 \le i,j \le n\}$ is linearly independent if and only if the mapping $y \mapsto w^*yw$ from A' into $L(H)$ is injective. The latter, however, implies the faithfulness of the subspace $\overline{wH}$ for A' (i.e. $y\overline{wH} = 0$ implies $y = 0$). Since $Sx = w^*xw$ (observe that on the right hand side x acts as a diagonal operator on K), it now follows from [12], thm 1.4.6 that $\{c_i^*c_j \mid 1 \le i,j \le n\}$ is linearly independent only if S is an extreme point in the set of all completely positive linear mappings T from $\tilde{A}$ into $L(H)$ with $T1 = c$. Here, $\tilde{A}$ is the C*-algebra generated by A and $1 \in L(H)$, and, obviously, $\tilde{A}' = A'$. Therefore, the linear independency of $\{c_i^*c_j \mid 1 \le i,j \le n\}$ is sufficient for S being an extreme point in $E\ell_{cp}(A;c)$.

To prove the necessity suppose that $\{c_i^*c_j \mid 1 \le i,j \le n\}$ is linearly dependent, thus $w^*yw = 0$ for some non-zero $y \in A'$. Replacing y by $\frac{y + y^*}{2\|y\|}$ we may assume that $y \in A_h'$ and $\|y\| \le 1$. Then $S = \frac{1}{2}(S_1 + S_2)$ where $S_1x = w^*x(1 + y)w$, $S_2x = w^*x(1 - y)w$. Since $1 \pm y \ge 0$, both S_1, S_2 are completely positive linear mappings from A into $\mathcal{L}(H)$, and also $S_11 = S_21 = c$. In addition, $(1 \pm y)^{\frac{1}{2}}$ correspond to positive $n\times n$-matrices $(\lambda_{ij})_{1\le i,j\le n}$ and $(\mu_{ij})_{1\le i,j\le n}$, respectively, thus by defining $a_i = \sum_{j=1}^{n} \lambda_{ij}c_j$ and $b_i = \sum_{j=1}^{n} \mu_{ij}c_j$ for $1 \le i \le n$ we obtain $S_1 = \sum_{i=1}^{n} M_{a_i^*,a_i}$, $S_2 = \sum_{i=1}^{n} M_{b_i^*,b_i}$ in $E\ell_{cp}(A;c)$. This shows that S is not an extreme point in $E\ell_{cp}(A;c)$.□

Corollary 5.6 was obtained for $A = M_n$ by Choi in [29], thm 5. We now strengthen the statement of Theorem 5.5 to a result on n-positive elementary operators.

Theorem 5.7. *Let A be a C^*-algebra and let S be an elementary operator on A of minimal length $n \ge 1$. If S is n-positive, then S is completely positive.*

Proof. First assume that A acts irreducibly on the Hilbert H and that S is represented as $S = \sum_{j=1}^{n} \lambda_j M_{b_j^*,b_j}$ with non-zero real numbers λ_j and a linearly independent subset $\{b_1,\dots,b_n\}$ of A like in Theorem 5.5.

From the proof of 5.5 we observe that $(\Lambda\xi|\xi) \ge 0$ for every $\xi \in \{\sum_{k=1}^{m} x_k v\xi_k \mid x_k \in A,\ \xi_k \in H,\ 1 \le m \le n\} =: K_o$ since S is n-positive. We show now that K_o is dense in K. Then Λ will be positive and thus S is completely positive.

Take $\xi_1 \in H$ such that $K_1 = \overline{Av\xi_1} \ne \{0\}$, and define p_1 as the orthogonal projection from K onto K_1. If $K_1 \ne K$, then,

since AvH is dense in K (cf. 5.5), there exists $\xi_2 \in H$ such that $K_2 = \overline{Av\xi_2} \not\subseteq K_1$. Define p_2 as the orthogonal projection from K onto $\overline{K_1 + K_2}$. If $\overline{K_1 + K_2} \neq K$, then there exists ξ_3 in H such that $K_3 = \overline{Av\xi_3} \not\subseteq \overline{K_1 + K_2}$, and we define p_3 analogously. By means of this, we obtain a strictly increasing sequence (p_k) of projections in $A' = \mathbb{C}1 \otimes M_n$. Therefore, $\sup_{1\leq k\leq n} p_k = 1$, i.e. $\overline{K_1 + \ldots + K_n} = \overline{K}_o = K$.

Now let A be arbitrary and π be an irreducible representation of A. Then S induces an n-positive elementary operator πS on $\pi(A)$ which has a representation of length $m \leq n$. We can assume that $\pi S = \sum_{j=1}^{m} M_{\pi(a_j),\pi(b_j)}$ with $a_j, b_j \in A$ and both $\{\pi(a_1),\ldots,\pi(a_m)\}$ and $\{\pi(b_1),\ldots,\pi(b_m)\}$ are linearly independent. By the above, πS is completely positive on $\pi(A)$. Since this holds for any irreducible representation, S is completely positive on A.□

This result extends the result for $n = 1$ proved in [110].

We conclude with a simple criterion that distinguishes $(n - 1)$-positivity from n-positivity for an elementary operator of minimal length n. Let A be a prime C*-algebra and let $S \in E\ell(A)$ with $\ell(S) = n > 1$. Suppose that S is $(n - 1)$-positive but not n-positive. Then, by Corollary 5.3, there exist a linearly independent set $\{b_1,\ldots,b_n\}$ in A and non-zero real numbers $\lambda_1,\ldots,\lambda_n$ such that $S = \sum_{j=1}^{n} \lambda_j M_{b_j^*,b_j}$. Since S is not n-positive there is some λ_j, say λ_n, which is negative. Therefore $\sum_{j=1}^{n-1} \lambda_j M_{b_j^*,b_j} \geq |\lambda_n| M_{b_n^*,b_n}$ in the order of $(n - 1)$-positive mappings in $\ell(A)$. Since $|\lambda_n| M_{b_n^*,b_n}$ is completely positive, $S_1 := \sum_{j=1}^{n-1} \lambda_j M_{b_j^*,b_j}$ is $(n - 1)$-positive, and since

$\ell(S_1) = n - 1$ by Proposition 4.7, S_1 is completely positive by Theorem 5.7 and therefore $\lambda_1,\ldots,\lambda_{n-1} > 0$.

Thus we established

Proposition 5.8. *Let* A *be a prime C*-algebra and let* $S \in E\ell(A)$ *with minimal length* $n > 1$. *If* S *is* $(n - 1)$*-positive but not* n*-positive then there exists a linearly independent subset* $\{c_1,\ldots,c_n\}$ *in* A *with the property that*

$$S = \sum_{j=1}^{n-1} M_{c_j^*,c_j} - M_{c_n^*,c_n}.$$

Observe that the conclusion in Proposition 5.8 is not a sufficient condition for $(n - 1)$-positivity. For example, the transposition $\tau \in E\ell(M_2)$ has minimal length $\ell(\tau) = 4$ and has a representation $\tau = \sum_{j=1}^{3} M_{c_j^*,c_j} - M_{c_4^*,c_4}$, but is only 1-positive (compare Example 5.4).

6. ELEMENTARY OPERATORS WHICH ARE COMPACT OR WEAKLY COMPACT

An element a of a Banach algebra A is called *compact* if the multiplication $M_{a,a}$ is a compact operator on A. We will denote the set of all compact elements in A by $K(A)$. This definition is due to Vala [144], while there are also other possible notions of compact elements of a Banach algebra (see [15], R.5 and Remark 6.13.3). Vala ([143], thm 3) proved that if $A = L(E)$ for a Banach space E then $K(A) = K(E)$, the ideal of all compact operators on E, and if $a,b \in L(E)$ are non-zero, then $M_{a,b}$ is compact if and only if $a,b \in K(E)$ (cf. also [20], § 33). This was exploited by Bonsall [18], [19] to develop an algebraic approach to the Riesz-Schauder theory of compact operators, e.g., Schauder's theorem is an immediate corollary ([20], 33.4). Fong and Sourour ([64], thm 2) extended Vala's result appropriately to elementary operators S on $L(E)$, and Ylinen [148], [149] and Akemann and Wright [2] characterized compactness and weak compactness of L_a, R_a, $L_a + R_a$, and $M_{a,a}$ where a is an element of a C*-algebra A. The latter authors used the same method to determine the compact and weakly compact derivations of A and also the *-automorphisms which are weakly compact perturbations of the identity ([1], [2], see also [141]).

This chapter is devoted to a generalization of the results obtained in [2] and [64] by means of the methods developed in the previous sections. In the sequel, we will confine ourselves to prime C*-algebras. The first reason is that, in an arbitrary

Banach algebra, $K(A)$ may fail to be an ideal ([144]). But even if $K(A)$ is an ideal, it need not be prime. For an example, take $A = L(E)$ where $E = L^1[0,1]$ and let $W(E)$ be the ideal of weakly compact operators on E. Then $W(E)^2 \subseteq K(E) \subsetneq W(E)$ ([49], VI.8.13), hence the Calkin algebra $C(E) = L(E)/K(E)$ is even not semi-prime. (An algebra is called *semi-prime* if it contains no non-zero nilpotent ideals.) However, the Calkin algebra $C(H)$ on a Hilbert space H is always prime as follows from Lemma 3.5. This will be the clue in our considerations.

We start our investigations by characterizing the weakly compact elementary operators on the algebra $L(H)$ as those which vanish on the Calkin algebra. Recall that a bounded linear operator T on a Banach space E is said to be *weakly compact* if T maps bounded subsets of E into relatively weakly compact subsets. We shall use the following equivalent formulation ([49], VI.4.2): T is weakly compact if and only if the second adjoint T^{**} maps the second dual E^{**} of E into the natural embedding ιE of E into E^{**}. Moreover, we note that the second dual $K(H)^{**}$ of the C*-algebra $K(H)$ of all compact operators on H can be identified with $L(H)$ ([136], II.1.8), and that an operator $T \in L(L(H))$ which leaves $K(H)$ invariant is ultraweakly continuous if and only if $(T_{|K(H)})^{**} = T$. For such T denote by $\tilde{T} \in L(C(H))$ the induced operator on the Calkin algebra. Our first lemma is a reformulation of [2], prop. 2.1.

Lemma 6.1. *Let $T: L(H) \to L(H)$ be an ultraweakly continuous bounded linear operator which leaves $K(H)$ invariant. Then T is weakly compact if and only if $\tilde{T} = 0$.*

Proof. If $T_1 = T_{|K(H)}$ denotes the restriction of T to $K(H)$ then, by Gantmacher's theorem ([49], VI.4.8), T_1 is weakly compact if and only if $T_1^{**} = T$ is weakly compact. By the above remarks, this is equivalent to $T_1^{**}L(H) \subseteq K(H)$, i.e. $T = 0$.□

Now, we are in a position to apply Theorem 4.2 in order to unify propositions 2.2, 2.3 and 2.7 of [2] (see also [148], thm 3.1 and [149], thm 3.1) and to generalize them to arbitrary elementary operators.

Proposition 6.2. *Let* $A = L(H)$, *and for* $a,b \in A^n$ *let* $S = \sum_{j=1}^{n} M_{a_j,b_j}$ *be the corresponding elementary operator. If the set* $\{b_1,\ldots,b_n\}$ *is linearly independent* mod $K(H)$ *then the following conditions are equivalent:*

(a) S *is weakly compact.*

(b) $a_j \in K(H)$ *for all* $1 \leq j \leq n$.

Proof. By Lemma 6.1, S is weakly compact if and only if the induced operator $\tilde{S}$ vanishes on the Calkin algebra. Denoting by π the canonical homomorphism from $L(H)$ onto $C(H)$ we have $\tilde{S} = \sum_{j=1}^{n} M_{\pi(a_j),\pi(b_j)}$. Since $C(H)$ is prime by Lemma 3.5, it follows from Theorem 4.2 that $\tilde{S} = 0$ if and only if $\pi(a_j) = 0$ for all j.□

In particular, 6.1 and 6.2 show that the primeness of $C(H)$ is nothing but the assertion $M_{a,b}$ is weakly compact if and only if a or b is compact. The following characterization of weakly compact derivations on $L(H)$ was first established by Akemann and Wright in [1], thm 3.1. We obtain it here as an immediate consequence.

Corollary 6.3. *The following conditions on a derivation* δ *on* $\mathcal{L}(H)$ *are equivalent:*

(a) δ *is weakly compact.*

(b) $\delta\mathcal{L}(H) \subseteq K(H)$.

(c) $\delta = \delta_a$ *for some* $a \in K(H)$.

Proof. (a) <=> (b) by 6.1 and (a) <=> (c) by 6.2.□

In order to extend Proposition 6.2 to prime C*-algebras we need two auxiliary results and some more information on $K(A)$.

Lemma 6.4. *Let* A *be a C*-algebra, let* π *be a representation of* A, *and for* $S \in \mathcal{EL}(A)$ *let* $\pi S \in \mathcal{EL}(\pi(A)'')$ *denote the ultraweakly continuous extension of* S *to the ultraweak closure* $\pi(A)''$ *of* $\pi(A)$. *If* S *is (weakly) compact, so is* πS *and then* $\pi S\pi(A)'' \subseteq \pi(A)$.

Proof. Denote by $\iota: A \to A^{**}$ and by $\tilde{\pi}: A^{**} \to \pi(A)''$ the canonical injection of A into its enveloping W*-algebra A^{**} and the unique extension of π to A^{**} respectively. Then $\tilde{\pi} \circ \iota = \pi$ and $\tilde{\pi}$ maps the closed unit ball A_1^{**} of A^{**} onto the closed unit ball $\pi(A)_1''$ of $\pi(A)''$ ([136], III.2.2). If $S = \sum_{j=1}^{n} M_{a_j,b_j}$ with $a_j, b_j \in A$ then $\pi S = \sum_{j=1}^{n} M_{\pi(a_j),\pi(b_j)'}$ hence

$$\pi S\tilde{\pi}(x) = \sum_{j=1}^{n} \pi(a_j)\tilde{\pi}(x)\pi(b_j) = \tilde{\pi}\Big(\sum_{j=1}^{n} \iota(a_j)x\iota(b_j)\Big) = \tilde{\pi}S^{**}x$$

for all $x \in A^{**}$. If S is (weakly) compact, so is its second adjoint S^{**} ([49], VI.4.8 and VI.5.2, respectively). From

$$\pi S\pi(A)_1'' = \pi S\tilde{\pi}(A_1^{**}) = \tilde{\pi}S^{**}A_1^{**} \subseteq \tilde{\pi} \circ \iota(A) = \pi(A)$$

we now obtain $\pi S\pi(A)'' \subseteq \pi(A)$ and the (weak) compactness of πS.□

Lemma 6.5. *Let* $S \in El(A)$ *for a C*-algebra* A. *If* S *is weakly compact then* $SA \subseteq K(A)$.

Proof. Let $\hat{A}$ denote the spectrum of A and let $\pi = \bigoplus_{t\in\hat{A}} \pi_t$ denote the reduced atomic representation of A. Then $\pi(A)'' = \sum_{t\in\hat{A}}^{\oplus} L(H_t)$ ([116], 4.3.8). If S is weakly compact on A, by Lemma 6.4, its extension πS is weakly compact on $\pi(A)''$. Denote by S_t the induced mapping on $\pi_t(A)'' = L(H_t)$ for each $t \in \hat{A}$. Since any compression of a weakly compact mapping is weakly compact, S_t is weakly compact and thus

(6.5.1) $\qquad S_t L(H_t) \subseteq K(H_t)$ by Lemma 6.1.

The next argument is taken from [1], lem. 3.2. We claim that, for each $\varepsilon > 0$, the set $\{t \in \hat{A} \mid \|S_t\| \geq \varepsilon\}$ is finite. If not, there exist a sequence $(t_n)_{n\in\mathbb{N}}$ in $\hat{A}$ and $x = \sum_{t\in\hat{A}}^{\oplus} x_t$ in $\pi(A)''$ such that $\|S_{t_n} x_{t_n}\| \geq 1$ for all $n \in \mathbb{N}$. Without restriction we may assume that πS acts on $\sum_{n\in\mathbb{N}}^{\oplus} L(H_{t_n})$, and we denote by B the restricted direct sum of $\{L(H_{t_n}) \mid n \in \mathbb{N}\}$, i.e.

$$B = \{(x_n)_{n\in\mathbb{N}} \in \sum_{n\in\mathbb{N}}^{\oplus} L(H_{t_n}) \mid \lim_{n\to\infty} \|x_n\| = 0\}.$$

Take a continuous linear functional ϕ which vanishes on B such that $\langle \phi, \pi S x\rangle = 1$. Define a sequence $(y^{(k)})_{k\in\mathbb{N}}$ in $\sum_{n\in\mathbb{N}}^{\oplus} L(H_{t_n})$ by $y_n^{(k)} = 0$ if $n < k$ and $y_n^{(k)} = x_{t_n}$ if $n \geq k$. Then $y^{(k)} \to 0$ in the weak operator topology, hence $\pi S y^{(k)} \to 0$ in the weak operator topology. Since the weak operator topology is a weaker Hausdorff topology than the weak topology, it coincides with the weak topology on the relatively weakly compact set $\{\pi S y^{(k)} \mid k \in \mathbb{N}\}$. Therefore $(\pi S y^{(k)})_{k\in\mathbb{N}}$ tends to zero weakly. But $\pi S y^{(k)} - \pi S x \in B$ for all $k \in \mathbb{N}$, thus $\langle \phi, \pi S y^{(k)}\rangle = \langle \phi, \pi S x\rangle = 1$, a contradiction. This proves the claim.

As a consequence, the set

$$\{t \in \hat{A} \mid \|S_t\| > 0\} = \bigcup_{n\in\mathbb{N}} \{t \in \hat{A} \mid \|S_t\| \geq \tfrac{1}{n}\}$$

is countable. Hence there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $\hat{A}$ such that $S_t = 0$ if $t \notin \{t_n \mid n \in \mathbb{N}\}$ and $\lim_{n\to\infty} \|S_{t_n}\| = 0$. Now if $y = \pi S x$ for some $x = \sum^{\oplus}_{t\in\hat{A}} x_t \in \pi(A)''$ then $y_t = 0$ for all $t \notin \{t_n \mid n \in \mathbb{N}\}$ and $\lim_{n\to\infty} \|y_{t_n}\| = 0$. From this, (6.5.1) and Lemma 6.4 we obtain

$$\pi S \sum^{\oplus}_{t\in\hat{A}} L(H_t) \subseteq \sum^{\oplus}_{t\in\hat{A}_0} K(H_t) \cap \pi(A), \tag{6.5.2}$$

where $\sum^{\oplus}_{t\in\hat{A}_0} K(H_t) =$

$$\{x = \sum^{\oplus}_{t\in\hat{A}} x_t \in \sum^{\oplus}_{t\in\hat{A}} L(H_t) \mid \forall\, \varepsilon > 0\colon \{t \in \hat{A} \mid \|x_t\| \geq \varepsilon\} \text{ is finite}\}$$

is the restricted direct sum of $\{K(H_t) \mid t \in \hat{A}\}$.

Our last claim is

$$\sum^{\oplus}_{t\in\hat{A}_0} K(H_t) \cap \pi(A) = \pi(K(A)). \tag{6.5.3}$$

This together with (6.5.2) will yield $\pi S \pi(A)'' \subseteq \pi(K(A))$ from which $SA \subseteq K(A)$ follows by the faithfulness of π ([116], 4.3.11).

In order to prove (6.5.3) note at first that, by definition, $M_{a,a}$ is compact for any $a \in K(A)$. Therefore, by (6.5.2), $M_{\pi(a),\pi(a)}\ \pi(A)'' \subseteq \sum^{\oplus}_{t\in\hat{A}_0} K(H_t)$. In particular, $\pi(a)\pi(a)^*\pi(a)$ belongs to $\sum^{\oplus}_{t\in\hat{A}_0} K(H_t)$, i.e. $\lim_{n\to\infty} \|\pi_{t_n}(a)\|^3 = 0$ for some sequence $(t_n)_{n\in\mathbb{N}}$ in $\hat{A}$ and $\|\pi_t(a)\|^3 = 0$ for all t which do not belong to $\{t_n \mid n \in \mathbb{N}\}$. Thus $\pi(a) \in \sum^{\oplus}_{t\in\hat{A}_0} K(H_t)$. Conversely, take $a \in A$ with $\pi(a) \in \sum^{\oplus}_{t\in\hat{A}_0} K(H_t)$, i.e. $\pi_t(a) = 0$ for all but a countable number of indices $t_n \in \hat{A}$, $\lim_{n\to\infty} \|\pi_{t_n}(a)\| = 0$ and $\pi_{t_n}(a) \in K(H_{t_n})$ for all $n \in \mathbb{N}$. By [20], 33.3 the

multiplication $M_{\pi_{t_n}(a),\pi_{t_n}(a)}$ is compact on $L(H_{t_n})$. Hence, $S_m = \bigoplus_{n=1}^{m} M_{\pi_{t_n}(a),\pi_{t_n}(a)}$ defines a sequence of compact mappings which converges uniformly to $M_{\pi(a),\pi(a)}$ since we have $\lim_{n\to\infty} \|M_{\pi_{t_n}(a),\pi_{t_n}(a)}\| = 0$. Thus $M_{a,a}$ is compact on A. The proof is complete.□

Now a detailed description of $K(A)$ follows. This is also related to the notion of minimal projections of A. A projection p of a C*-algebra A is called *minimal* if $M_{p,p}$ is a rank-one projection on A. A *minimal left ideal* L of A is a non-zero left ideal which contains no other left ideals than 0 and L. By [15], BA.3.1 and BA.4.3, a left ideal is minimal if and only if $L = Ap$ for some minimal projection $p \in A$. If A contains minimal left ideals, the sum of all of them is called the *socle*, $\mathrm{soc}(A)$, of A (cf. [20], 31.10). Equivalently, $\mathrm{soc}(A)$ is the ideal generated by all minimal projections of A. If A contains no minimal left ideals, we put $\mathrm{soc}(A) = 0$. In any case, $\mathrm{soc}(A)$ is the set of all *finite rank elements* of A, i.e. $a \in \mathrm{soc}(A)$ if and only if $M_{a,a}$ is of finite rank ([15], C*.1.2).

The next proposition lists several known characterizations of the compact elements of a C*-algebra A. The equivalences (b) <=> (c) <=> (e) are implicit in [2], and (b) <=> (c) is [148], thm 3.1. Here, they follow as a consequence of Lemma 6.5. A C*-algebra is said to be of *elementary type* if it is isomorphic to $K(H)$ for some Hilbert space H.

Proposition 6.6. *The following conditions on an element* a *of a C*-algebra* A *are equivalent:*

(a) $a \in K(A)$.

(b) $M_{a,a}$ *is compact.*

(c) $M_{a,a}$ *is weakly compact.*

(d) $a \in \overline{soc(A)}$.

(e) $a \in \sum^{\oplus}_{n\in\mathbb{N}^0} I_n$ *where* $(I_n)_{n\in\mathbb{N}}$ *is a sequence of orthogonal closed ideals of* A *of elementary type.*

Proof. (a) <=> (b) by definition. (b) <=> (d) by [15], C*.1.3. (b) => (c) is clear. (c) => (a) by Lemma 6.5, since it follows from the hypothesis that $aAa \subseteq K(A)$ and $K(A)$ is a closed ideal by (d).

(a) <=> (e) by (6.5.3) since $\pi_{t_n}(a) \in K(H_{t_n}) \cap \pi_{t_n}(A) \neq 0$ for $a \neq 0$ (we use the same notation as in (6.5.3)). By [116], 6.1.4, this implies $K(H_{t_n}) \subseteq \pi_{t_n}(A)$. Hence we may define $I_n = \pi^{-1}(K(H_{t_n}))$ for each $n \in \mathbb{N}$.□

In particular, $K(A)$ is a closed ideal of the C*-algebra A. Therefore, Lemma 6.5 together with Lemma 6.1 suggest a generalization of Proposition 6.2 to arbitrary C*-algebras. However, this fails as is demonstrated by the following counterexample.

Example 6.7. Let $A = L(H) \oplus L(H)$ for an infinite dimensional Hilbert space H. Then $K(A) = K(H) \oplus K(H)$ and $K(A)^{**} = K(H)^{**} \oplus K(H)^{**} = A$. If $c_1, c_2 \in K(H)$ and $a = c_1 \oplus 1$, $b = 1 \oplus c_2$ then neither a nor b belong to $K(A)$ while $a(x_1 \oplus x_2)b = c_1x_1 \oplus x_2c_2$ belongs to $K(A)$ for all x_1, x_2 in $L(H)$. Hence $M_{a,b}A \subseteq K(A)$ so that $M_{a,b}$ is weakly compact and $K(A)$ is not a prime ideal.

The difficulties arising from the existence of non-zero orthogonal ideals can be overcome in prime C*-algebras. It will follow from 6.6 (see below) that, in a prime C*-algebra, $K(A)$ is isomorphic to $K(H)$ via an irreducible representation (π,H) of A unless $K(A) = 0$. On the other hand, if one reads the assertion of 6.2 for a C*-algebra A with $K(A) = 0$, one realizes that the primeness of A is a necessary condition for such an assertion, too.

In the following, we have to distinguish between prime C*-algebras with $K(A) = 0$ and those which contain minimal projections. It turns out that the latter are already primitive. To this end, suppose that A is a prime C*-algebra with $K(A) \neq 0$. Then, by Proposition 6.6.(d), A contains minimal projections. For any minimal projection $p \in A$ endow the minimal left ideal Ap with the inner product

$$\langle x,y\rangle := y^*x \qquad (x,y \in Ap)$$

and write H_p for $(Ap,\langle\cdot,\cdot\rangle)$. Then, H_p is a Hilbert space whose inner product norm coincides with the original norm on Ap ([15], p.79). Define a representation π_p of A on H_p by $\pi_p(a)x = ax$ $(a \in A,\ x \in H_p)$. Since Ap is minimal, π_p is irreducible, and since $\pi_p(a) = 0$ implies $aAp = 0$ and A is prime, π_p is also faithful. Hence, A is a primitive C*-algebra. In addition, by [15], F.4.3, $\pi_p(\mathrm{soc}(A)) = F(H_p)$, the finite rank operators on H_p, and $\pi_p(K(A)) = K(H_p)$. Also $\pi_p(p) = p \otimes p$ is a minimal projection in $L(H_p)$ ([15], C*.4.2, proof).

In fact, as can be seen easily, π_p is nothing but the GNS-representation of A associated with the pure state ϕ_p defined by $\langle\phi_p,a\rangle := pap$ $(a \in A)$, because $A(1 - p) = \{x \in A \mid \langle\phi_p,x^*x\rangle = 0\}$ and p is the support projection of ϕ_p.

Examples of prime C*-algebras with $K(A) = 0$ are given by the so-called *Glimm algebras* ([116], 6.5.7). Generally, a prime C*-algebra contains non-zero compact elements if and only if it is not antiliminal (see 6.14.2).

With these preparations it is now easy to characterize the weakly compact elementary operators on a prime C*-algebra.

Theorem 6.8. *Let* A *be a prime C*-algebra and for* $a,b \in A^n$ *let* $S = \sum_{j=1}^{n} M_{a_j,b_j}$ *be the corresponding elementary operator. If the set* $\{b_1,\ldots,b_n\}$ *is linearly independent* mod $K(A)$ *then the following conditions are equivalent:*

(a) S *is weakly compact.*

(b) $a_j \in K(A)$ *for all* $1 \leq j \leq n$.

Proof. (b) => (a) This is true in any C*-algebra and without the additional assumption on $\{b_1,\ldots,b_n\}$. If $a_j \in K(A)$ for all j then L_{a_j} is weakly compact for all j ([2], thm 2.5; the proof is essentially the second argument for (6.5.3) together with the corresponding statement for $A = L(H)$). Since the weakly compact operators form an ideal ([49], VI.4.6) the operator $S = \sum_{j=1}^{n} L_{a_j} R_{b_j}$ is weakly compact.

(a) => (b) First assume that $K(A) = 0$. Then, by Lemma 6.5, S is weakly compact if and only if $S = 0$. Hence the assertion follows by Theorem 4.2.

Now assume $K(A) \neq 0$. Then $K(A) = \pi_p^{-1}(K(H_p))$ for the faithful irreducible representation π_p corresponding to some minimal projection p of A as defined above. Since the set $\{\pi_p(b_1),\ldots,\pi_p(b_n)\}$ is linearly independent mod $K(H_p)$ and $\pi S = \sum_{j=1}^{n} M_{\pi_p(a_j),\pi_p(b_j)}$ is weakly compact on $L(H_p)$ by Lemma

6.4, we obtain $\pi_p(a_j) \in K(H_p)$ by Proposition 6.2. Thus a_j belongs to $K(A)$ for all $1 \leq j \leq n$.□

So far we have concentrated on the weak compactness of elementary operators as this is apparently more closely related to primeness (of the Calkin algebra) than compactness. Now we shall add a characterization of compact elementary operators on prime C*-algebras without extra effort. The following was proved for $A = L(E)$ by Fong and Sourour ([64], thm 2).

Theorem 6.9. *Let* A *be a prime C*-algebra and for* $a,b \in A^n$ *let* $S = \sum_{j=1}^{n} M_{a_j,b_j}$ *be the corresponding elementary operator.*

(i) *If* S *is compact and* $\{b_1,\ldots,b_n\}$ *is linearly independent, then* $a_j \in K(A)$ *for all* $1 \leq j \leq n$.

(ii) S *is compact if and only if there are* $c_1,\ldots,c_n$ *in* $\text{lin}\{a_1,\ldots,a_n\} \cap K(A)$ *and* $d_1,\ldots,d_n$ *in* $\text{lin}\{b_1,\ldots,b_n\} \cap K(A)$ *such that* $S = \sum_{j=1}^{n} M_{c_j,d_j}$.

Proof. If $K(A) = 0$, then S is compact if and only if $S = 0$ by Lemma 6.5. Hence, assertion (i) is a consequence of Theorem 4.2 while (ii) is trivial.

If $K(A) \neq 0$, then $K(A) = \pi^{-1}(K(H))$ for some faithful irreducible representation (π,H) of A. Since S is compact if and only if the induced elementary operator $\pi S \in E\ell(L(H))$ is compact (Lemma 6.4), (i) and (ii) follow from the corresponding result on $L(H)$ ([64], thm 2).□

Alternatively, the "only if"-part of (ii) can be deduced directly from (i) by considering a minimal representation of S (cf. Proposition 4.7 and p.49).

In analogy to Corollary 6.3, the following characterization of compact derivations on $L(H)$ which can be found in

[87] is derived from Theorem 6.9.

Corollary 6.10. *If* H *is an infinite dimensional Hilbert space, then there are no non-zero compact derivations on* $L(H)$.

Proof. If $\delta(x) = ax - xa$ is a compact derivation on $L(H)$ and $\delta \neq 0$, then $\{1,a\}$ is linearly independent. Thus, by 6.9.(i), both a and 1 are compact, i.e. dim H < ∞.□

Evidently, both Corollary 6.3 and 6.10 hold for every (inner) derivation on an infinite dimensional prime C*-algebra.

The next result was obtained for primitive C*-algebras by Ylinen ([147], cor. 4 on p.16).

Corollary 6.11. *Let* A *be a prime C*-algebra and let* $a,b \in A \setminus \{0\}$. *Then* $M_{a,b}$ *is compact if and only if both* a *and* b *belong to* K(A).

Proof. This follows immediately from Theorem 6.9 and the fact that the mapping $x \to M_{b^*,a^*}x = (ax^*b)^*$ is compact if $M_{a,b}$ is compact.□

In [64] Fong and Sourour conjectured that on the Calkin algebra C(H) on a separable Hilbert space H there cannot exist any non-zero compact elementary operator. This conjecture was confirmed by Apostol and Fialkow [7], and independently by Magajna[105]. However, much more is true.

Corollary 6.12. *Let* A *be a C*-algebra without non-zero compact elements. Then there are no weakly compact elementary operators on* A *different from zero.*

Proof. This is immediate from Lemma 6.5.□

Since, for any Hilbert space H, C(H) does not contain non-zero compact elements (6.14.5 and 6.14.2), Corollary 6.12

provides another proof for the conjecture by Fong and Sourour.

We conclude this chapter with some remarks and a supplement containing additional properties of $K(A)$.

Remarks 6.13. 1. Both in Theorem 6.8 and 6.9.(i) the assumption "$\{b_1,\dots,b_n\}$ is linearly independent (mod $K(A)$)" can be replaced by "$\{b_1,\dots,b_m\}$ is a maximal linearly independent subset (mod $K(A)$) of $\{b_1,\dots,b_n\}$" with the obvious alterations in the conclusions (compare Theorem 4.2).

2. The (weak) compactness of an elementary operator S on A is related to the range inclusion $SA \subseteq K(A)$. Other inclusions of this kind, where $A = \mathcal{L}(H)$ and SA is contained in some ideal of $\mathcal{L}(H)$, were investigated by Apostol and Fialkow [7] and Fialkow and Loebl[62].

3. There is a different concept of compact elements in von Neumann algebras which was introduced by Breuer ([21], [22]) and was from 1977 on elaborated mainly by Kaftal (see [94], the references in [70] and [27], see also [115]). If A is a von Neumann algebra, let $m(A)$ denote the closed ideal generated by the finite projections (finite in the sense of Murray-von Neumann). An element $a \in m(A)$ is called *compact relative to* A. This definition also extends the classical situation $A = \mathcal{L}(H)$ and allows a rather satisfactory theory of compact and Fredholm elements in von Neumann algebras, at least in the semifinite case. Also, an index group can be defined, and several interesting applications of this approach are known (see e.g. [24], 6.5 and 6.6). However, since $K(A)$ might be strictly smaller than $m(A)$, e.g. in type II von Neumann algebras, in view of 6.11 (and the preceding results) $m(A)$ is of much less importance for our purposes.

Supplement 6.14. Some properties of $K(A)$.

In this section we establish some additional properties of the ideal $K(A)$ of compact elements in a C*-algebra A. In the classical case $A = L(H)$, $K(A)$ coincides with $K(H)$ and is therefore the only minimal closed ideal of A. This, of course, is not true for arbitrary C*-algebras (e.g. see 6.7).

1. *If* A *is a prime C*-algebra then* $\mathrm{soc}(A)$ *is contained in every non-zero closed ideal of* A. *Thus, if* $K(A) \neq 0$, *it is the only minimal closed ideal of* A.

Proof. Let e be a minimal projection in A. Then $P_e = \{x \in A \mid xAe = 0\}$ is the unique primitive ideal in A which does not contain e ([15], BA.3.5). Since A is prime $P_e = 0$. Thus every non-zero primitive ideal P of A contains e, and therefore $\mathrm{soc}(A) \subseteq P$. Since each closed ideal of A is the intersection of those primitive ideals which contain it ([116], 3.13.8), $\mathrm{soc}(A)$ is contained in every non-zero closed ideal. Since $K(A) = \overline{\mathrm{soc}(A)}$ by 6.6, the second assertion follows.□

We now investigate the structure of $K(A)$ for a prime C*-algebra A. First recall some definitions from [116], chap. 6 and [46], § 4. A positive element a in a C*-algebra A is called *abelian* if the range of the multiplication $M_{a,a}$ is a commutative algebra. If A is generated as a C*-algebra by its abelian elements it is said to be *of type* I_o. Each C*-algebra contains a largest ideal of type I_o, namely the C*-subalgebra I_{abel} generated by all abelian elements ([116], 6.1.7). A C*-algebra A is called *liminal* if $\pi(A) = K(H)$ for each irreducible representation (π, H) of A. For example, each C*-algebra of type I_o is liminal.

The set

$$I_{lim} = \{x \in A \mid \pi(x) \in K(H) \text{ for all irreducible representations } (\pi,H) \text{ of } A\}$$

is the largest liminal ideal of the C*-algebra A ([116], 6.2.2). A C*-algebra A is called *antiliminal* if $I_{lim} = 0$. This amounts to the same as demanding $I_{abel} = 0$ ([116], 6.2).

2. *If A is a prime C*-algebra then* $K(A) = I_{abel}$. *In particular, a prime C*-algebra A is antiliminal if and only if* $K(A) = 0$.

Proof. If A does not contain any non-zero abelian element then there is no minimal projection in A whence $K(A) = I_{abel} = 0$. Thus we may assume that A is not antiliminal. Then, by Observation 1 above, $K(A) \subseteq I_{abel}$. To prove the reverse inclusion take a non-zero abelian element a in A.

Then

$$axaaya = ayaaxa \qquad \text{for all } x,y \in A.$$

Thus $(a^2xa^2)ya^2 - a^2y(a^2xa^2) = 0$ for all $x,y \in A$. Since A is prime, by 4.2 for each $x \in A$ there is $\lambda(x) \in \mathbb{C}$ such that $a^2xa^2 = \lambda(x)a^2$. In particular, $M_{a^2,a^2A} \subseteq \mathbb{C}a^2$. Since $a^6 = \lambda(a^2)a^2$ with $\lambda(a^2) > 0$ the element $e := \lambda(a^2)^{-1}a^4$ is a projection in A. Moreover

$$eAe = \lambda(a^2)^{-2}a^4Aa^4 \subseteq \mathbb{C}\lambda(a^2)^{-1}a^4 = \mathbb{C}e$$

shows that e is minimal. Therefore, $a = \lambda(a^2)^{1/4}e$ is a scalar multiple of a minimal projection which proves that $I_{abel} \subseteq K(A)$. □

It follows from the above proof that a prime C*-algebra which is not antiliminal is already primitive (compare [9], lem. 3.1). Our approach via elementary operators reveals that this

is only a special case of Martindale's characterization of prime rings satisfying a generalized polynomial identity ([82], 1.3.2).

The inclusion $K(A) \subseteq I_{abel}$ remains true without primeness.

3. *For every C*-algebra* A *the ideal* $K(A)$ *is of type* I_o.

Proof. We may assume that A acts on a Hilbert space H such that $soc(A) = A \cap F(H)$ where $F(H) = soc(L(H))$ is the ideal of finite rank operators on H and that every minimal projection in A is also minimal in $L(H)$ ([15], C*.4.3). For each non-zero $x \in soc(A)$ there are $a_j \in A$ and minimal projections $e_j \in A$, $1 \leq j \leq n$, such that $x = \sum_{j=1}^{n} a_j e_j$. Since $e_j = \xi_j \otimes \xi_j$ for some unit vectors $\xi_j \in H$ we have $a_j e_j = a_j \xi_j \otimes \xi_j$. For every $a \in A$ and every minimal projection $e = \xi \otimes \xi \in soc(A)$ the identity

$$a\xi \otimes \xi = \frac{1}{2}((a\xi + \xi)\otimes(a\xi + \xi) - i(ia\xi + \xi)\otimes(ia\xi + \xi) - a\xi \otimes a\xi - \xi \otimes \xi - ia\xi \otimes a\xi - i\xi \otimes \xi)$$

holds. Each term belongs to A, e.g.

$$(a\xi + \xi)\otimes(a\xi + \xi) = aea^* + ae + ea^* + e \in A,$$

and each is a multiple of a minimal projection in $L(H)$, e.g.

$$(a\xi + \xi)\otimes(a\xi + \xi) = \|a\xi + \xi\|^2 p_{a\xi + \xi},$$

where $p_{a\xi + \xi}$ is the orthogonal projection onto $\lin\{a\xi + \xi\}$. Hence each $a_j e_j$ is a linear combination of minimal projections in A. Since minimal projections are clearly abelian, x is a linear combination of abelian elements in A. Since $K(A) = \overline{soc(A)}$, it is generated as a C*-algebra by its abelian elements and thus of type I_o. □

If X is a compact Hausdorff space without isolated points the algebra $C(X)$ of continuous complex-valued functions on X provides an example of a type I_o C*-algebra with $K(A) = 0$. Since $K(A)$ is of type I_o, it is liminal and thus contained in I_{lim} for every C*-algebra A. (This inclusion was already observed by Ylinen [147], p.32.)

4. *If* A *is a prime C*-algebra then* $K(A) = I_{lim}$.

Proof. If A is antiliminal then $K(A) = I_{lim} = 0$. Otherwise, A possesses a faithful irreducible representation (ρ, H_ρ) such that $\rho(K(A)) = K(H_\rho)$ (see p.79). Thus $I_{lim} = \bigcap\{\pi^{-1}(K(H_\pi)) \mid (\pi, H_\pi)$ is an irreducible representation of $A\} \subseteq \rho^{-1}(K(H_\rho)) = K(A)$. □

Finally we ask when non-zero compact elements can exist in $A_{/K(A)}$ for a prime C*-algebra A. There are two trivial cases. Firstly, if A is antiliminal then $K(A) = 0$ hence $K(A_{/K(A)}) = 0$. Secondly, if A is liminal then $K(A) = A$ hence $K(A_{/K(A)}) = 0$ again. But there are still another classes of C*-algebras. A C*-algebra A is called *postliminal* or *of type I* if each non-zero quotient of A contains a non-zero abelian element. Thus, in a postliminal C*-algebra which is not liminal we may have $K(A_{/K(A)}) \neq 0$. An example is provided by the C*-algebra A generated by $K(H)$ and the identity on an infinite dimensional Hilbert space H. In this case, $K(A) = K(H)$ and $K(A_{/K(A)}) = \mathbb{C}1$. The answer is not clear for primitive C*-algebras that are neither antiliminal nor of type I. One of those algebras is $L(H)$ in which case the answer is, of course, well known.

5. *For every Hilbert space* H *the Calkin algebra* $C(H)$ *is antiliminal.*

Proof. By 6.14.2 it suffices to show that $K(C(H)) = 0$. Take $x \in L(H)$ such that $\pi(x) = \pi(x)^* \neq 0$ where $\pi: L(H) \to C(H)$ is the canonical quotient map. Replacing x by $\mathrm{Re}(x)$ we may assume x to be self-adjoint. Put $a = x^2$ and let p be an infinite dimensional projection such that $ap = pa \geq \varepsilon p$ for some $\varepsilon > 0$ (compare 3.5). Then there is $a' \in L(H)_+$ such that $aa' = a'a = p$. If $x_n = a'p_n a'$ where $(p_n)_{n\in\mathbb{N}}$ is a sequence of infinite dimensional mutually orthogonal projections with sum p, then $(\pi(x_n))_{n\in\mathbb{N}}$ is a bounded sequence in $C(H)$ with the property that

$$\|\pi(a)\pi(x_n)\pi(a) - \pi(a)\pi(x_m)\pi(a)\| = \|\pi(p_n) - \pi(p_m)\| = 1$$

for all $n \neq m$. By means of this, $M_{\pi(a),\pi(a)} = M_{\pi(x),\pi(x)}{}^2$ is not compact on $C(H)$, and therefore $\pi(x) \notin K(C(H))$. □

Since a factor of type II or type III does not contain nonzero compact elements, $K(A_{/K(A)}) = 0$ actually holds for any factor A. This shows that in a factor A the ideal $K(A)$ is also the largest postliminal ideal of A (cf. [46], 4.3.6).

7. THE SPECTRA OF ELEMENTARY OPERATORS

The spectral theory has without doubt been that part of the theory of elementary operators with the most active research. It is therefore impossible to treat all its aspects here, and our bibliography on this subject is far from being complete. Apart from the fundamental paper by Lumer and Roserblum [102], the most important contributions were made by Curto, Fialkow and Harte. However, nearly all authors have concentrated on elementary operators defined on $L(H)$, on $C(H)$, or on other very special types of algebras. As a result, the analysis has become very subtle and profound and at the same time the techniques more and more involved. In this last chapter the concept of primeness will again prove as an appropriate tool to extend many of the known results to much more general algebras, sometimes to the most general extent. At the same time, proofs will become simple and perspicuous.

It has emerged that for describing the spectra of elementary operators, a notion of 'joint spectrum' is inevitable. Among many possibilities ([124]) we have choosen the one introduced by R.E.Harte [71] which has the advantage that it can be easily defined for non-commuting n-tuples of elements in a normed algebra. The definition and some basic properties of this joint spectrum are compiled in Appendix 2.

Very interesting work on the spectra of elementary operators has been done by R.E.Curto by using Taylor's joint spectrum ([138]; [37], [40]; see also [36], [39]). His results,

mainly for the algebra $L(H)$, cannot be compared directly with ours, although there are connections between the Harte and the Taylor spectrum ([74], [38]).

Our first result describes invertibility of multiplications on an arbitrary unital ring. This generality is somewhat surprising when compared with the more elaborated techniques in the literature.

Proposition 7.1. *Let* a,b *be elements in a unital ring* R.

(i) *The multiplication* $M_{a,b}$ *is invertible if and only if both* a *and* b *are invertible.*

(ii) *The multiplication* $M_{a,b}$ *is left invertible if* a *is left invertible and* b *is right invertible.*

(iii) *The multiplication* $M_{a,b}$ *is right invertible if and only if* a *is right invertible and* b *is left invertible.*

Proof. We note that, if a is left invertible, then L_a is left invertible and R_a is right invertible, and if a is right invertible, then L_a is right invertible and R_a is left invertible. Thus (ii) is immediate, and so is the "if"-part of (iii). If, however, $1 = M_{a,b}T = L_a(R_bT) = R_b(L_aT)$ for some additive map T on R, then both L_a and R_b are right invertible. It follows that a is right invertible and b is left invertible which completes the proof of (iii).

If L_a is invertible, say $TL_a = L_aT = 1$, then $a \cdot T1 = 1$ thus $L_aL_{T1} = 1$. Therefore, $T = L_{T1}$ and $L_{T1}L_a1 = T1 \cdot a = 1$ yields that a is left invertible, too. A similar argument holds for R_a, and we end up with a invertible iff L_a invertible iff R_a invertible. Using A.2.4 we now have in succession:

$M_{a,b}$ invertible iff $\{L_aR_b, R_bL_a\} \subseteq \mathrm{Inv}(R^R)$ iff $\{L_a, R_b\} \subseteq \mathrm{Inv}(R^R)$ iff $\{a,b\} \subseteq \mathrm{Inv}(R)$. This proves (i). □

Remark. In our view, the spectral theory of elementary operators ought to be as easy as the proof of Proposition 7.1. This is not justified by the literature, but the remainder of this chapter may support our opinion. The missing equivalence in 7.1.(ii) is intrinsic to the possible left-right asymmetry in rings and general Banach algebras. It can be overcome only in special cases, e.g. in C*-algebras. A similar effect occurs in our next result.

Corollary 7.2. *Let* A *be a unital normed algebra, and let* $a,b \in A$.

(i) *The conditions*

(a) $M_{a,b}$ *is left invertible.*

(b) $M_{a,b}$ *is bounded below.*

(c) a *is not a left topological divisor of zero and* b *is not a right topological divisor of zero.*

satisfy (a) => (b) <=> (c).

(ii) *The conditions*

(a) $M_{a,b}$ *is right invertible.*

(b) $M_{a,b}$ *is surjective.*

(c) a *is not a right topological divisor of zero and* b *is not a left topological divisor of zero.*

satisfy (a) <=> (b) => (c).

Proof. ad (i). (a) => (b) is trivial.

(b) => (c) If $M_{a,b}$ is bounded below there is a constant $k > 0$ such that for all $x \in A$

$$k\|x\| \leq \|M_{a,b}x\| \leq \|ax\|\,\|b\|, \quad \|a\|\,\|xb\|$$

from which the assertion follows.

(c) => (b) Suppose that $M_{a,b}$ is not bounded below and $(x_n)_{n\in\mathbb{N}}$ is a sequence such that $\|x_n\| = 1$ for all n and $(axb)_{n\in\mathbb{N}}$ tends to zero. If $ax_n \to 0$ then a is a left topological divisor of zero. Otherwise, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ with $\inf_k \|ax_{n_k}\| > 0$, and by putting $y_k = ax_{n_k}/\|ax_{n_k}\|$ for all $k \in \mathbb{N}$ we obtain a sequence of unit vectors such that $y_k b \to 0$. Thus b is a right topological divisor of zero.

ad (ii). (a) => (b) is obvious, and (a) => (c) follows from 7.1.(iii).

(b) => (a) If $M_{a,b}$ is surjective there is $x \in A$ with $axb = 1$, hence a is right invertible and b is left invertible. The assertion now follows by 7.1.(iii). □

Corollary 7.3. *Let* a,b *be elements in a unital C*-algebra. Then all conditions in* 7.2.(i) *are equivalent, and all conditions in* 7.2.(ii) *are equivalent.*

Proof. As a special case of A.2.6, a is left invertible iff a is not a left topological divisor of zero and b is right invertible iff b is not a right topological divisor of zero. Thus (c) => (a) in 7.2.(i) because of 7.1.(ii).
Similarly (c) => (a) in 7.2.(ii). □

A proof of Corollary 7.3 for the case $A = L(H)$ can be found at various places ([60], thms 2.3 and 2.8; [145], thms 8 and 9). The paper of Weber [145] also contains the next result for $A = L(H)$ and $A = C(H)$ (ibid., cor. on p.740 and lem. 7).

Proposition 7.4. *A multiplication on a unital Banach algebra is surjective if and only if it has dense range.*

Proof. If $a,b \in A$ and $A = \overline{M_{a,b}A}$ then $A \subseteq \overline{aA} \cap \overline{Ab}$. Hence, the right ideal generated by a and the left ideal generated by b are dense in A, therefore equal to A since A is unital. Taking $a',b' \in A$ with $aa' = b'b = 1$ we obtain $M_{a,b} M_{a',b'} = id_A$ and arrive at the assertion. □

Remark. An argument similar to 7.2.(i), (b) <=> (c), but simpler shows that a multiplication $M_{a,b}$ on a ring R is injective if and only if a is not a left divisor of zero and b is not a right divisor of zero.

So far we merely considered the spectral value 0 of a multiplication. The fairly general results show that 0 must play a somewhat exceptional rôle. This will be observed later again. We now turn our attention to the non-zero spectral values. As already indicated in Proposition 3.1, the primeness of A will be essential.

Proposition 7.5. *Let* A *be a prime algebra with identity, let* $a,b \in A^n$ *and* $S = \sum_{j=1}^{n} M_{a_j,b_j}$ *be the corresponding elementary operator. The following inclusions are valid*

$$\{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in P\sigma(a),\ \mu \in P\sigma_r(b) \} \subseteq P\sigma(S)$$

and

$$\{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in P\sigma_r(a),\ \mu \in P\sigma(b) \} \subseteq P\sigma_r(S).$$

Proof. Suppose that x and y are non-zero elements in A such that for some $\lambda, \mu \in \mathbb{C}^n$ we have $(\lambda_j - a_j)x = 0 = y(\mu_j - b_j)$ for all j. By hypothesis, the multiplication $M_{x,y}$ is non-zero. By multiplying the identity

$$\sum_{j=1}^{n} \lambda_j \mu_j - S = \sum_{j=1}^{n} (M_{\lambda_j - a_j, \mu_j} + M_{a_j, \mu_j - b_j})$$

on the right by $M_{x,y}$, we thus obtain the first assertion. The second can be proved in the same manner. □

From now on we will assume that A is a unital Banach algebra. The reader will have no difficulty to decide to which extent the completeness of A is dispensable.

As an immediate consequence of Proposition 7.5 and A.2.9 we obtain an inclusion for the approximate point spectra.

Corollary 7.6. *Let* A *be an ultraprime unital Banach algebra, let* $a,b \in A^n$ *and* $S = \sum_{j=1}^{n} M_{a_j,b_j}$ *be the corresponding elementary operator. Then*

$$\{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in AP\sigma(a),\ \mu \in AP\sigma_r(b) \} \subseteq AP\sigma(S)$$

and

$$\{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in AP\sigma_r(a),\ \mu \in AP\sigma(b) \} \subseteq AP\sigma_r(S).$$

Proof. We have

$$\{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in AP\sigma_r(a),\ \mu \in AP\sigma(b) \}$$

$$= \{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in P\sigma_r(\hat{a}),\ \mu \in P\sigma(\hat{b}) \} \qquad \text{by A.2.9}$$

$$\subseteq P\sigma_r(\hat{S}) \qquad \text{by 7.5}$$

$$= AP\sigma_r(S) \qquad \text{by A.2.9}$$

since $\hat{S} = \sum_{j=1}^{n} M_{\hat{a}_j,\hat{b}_j} \in E\ell(\hat{A}_U)$ where $\hat{A}_U$ is some prime ultrapower of A. This proves the second assertion, and the first is obtained analogously. □

Corollary 7.6 follows in the case $A = L(E)$ from [71], thm 3.2 and [72], thm 3.2. It is easy to see that

$$0 \in AP\sigma(a) \cup AP\sigma_r(b) \Rightarrow 0 \in AP\sigma(S)$$

and

$$0 \in AP\sigma_r(a) \cup AP\sigma(b) \Rightarrow 0 \in AP\sigma_r(S)$$

is valid in every unital Banach algebra.

For a reversal of any of the above spectral inclusions one has to impose some commutativity assumptions on a and b. Easy examples on $A = L(H)$ show that both inclusions in 7.5 can be strict. Combining [60], thm 2.5 and Proposition 7.15 (see below) one obtains

$$P\sigma_r(S) = P\sigma_r(S_/) \cup P\sigma_r(S_|)$$

where $S \in El(L(H))$ is induced by commuting n-tuples and $S_/$ resp. $S_|$ are the operator induced by S on the Calkin algebra $C(H)$ resp. the restriction of S to the compact operators $K(H)$.[1] Very little information is available on the point spectrum of S; a characterization of injectivity of $\delta_{a,b}$ on $L(H)$ is given in [57], prop. 4.1. Some 'upper bounds' for $P\sigma(S)$ and $P\sigma_r(S)$ are discussed in [111], 3.14 and 3.15. The other parts of the spectrum of S however can be described more precisely. To reach this aim we need another lemma.

Lemma 7.7. *Let* A *be an ultraprime unital Banach algebra. If* $S = \sum_{j=1}^{n} M_{a_j,b_j}$ *with commuting n-tuples* $a,b \in A^n$ *then the set*

$$\{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in AP\sigma(a),\ \mu \in AP\sigma(b) \text{ or } \lambda \in AP\sigma_r(a),\ \mu \in AP\sigma_r(b) \}$$

is contained in $\sigma(S)$.

Proof. By using a prime ultrapower of A and A.2.9 as in the proof of 7.6 it suffices to prove the assertion for the

[1] An n-tuple $a \in A^n$ is called *commuting* if all a_j commute.

point spectra. Suppose $\lambda \in P\sigma(a)$, $\mu \in P\sigma(b)$ and $\rho = \sum_{j=1}^{n} \lambda_j \mu_j$ does not belong to $\sigma(S)$. If $x,y \in A \setminus \{0\}$ are such that $(\lambda_j - a_j)x = (\mu_j - b_j)y = 0$ for all $1 \le j \le n$, we take z,u in A such that $xzy \neq 0$ and $(\rho - S)u = xz$. Since

$$(\rho - S)(\lambda_j - a_j)u = (\lambda_j - a_j)(\rho - S)u = (\lambda_j - a_j)xz = 0$$

and $\rho - S$ is injective, it follows that $(\lambda_j - a_j)u = 0$ for all j. Thus

$$\begin{aligned} xzy &= ((\rho - S)u)y = \Big(\sum_{j=1}^{n} (\lambda_j \mu_j - M_{a_j,b_j})u\Big)y \\ &= \Big(\sum_{j=1}^{n} (M_{\lambda_j - a_j, b_j} + M_{\lambda_j, \mu_j - b_j})u\Big)y \\ &= \sum_{j=1}^{n} ((\lambda_j - a_j)ub_j y + \lambda_j u(\mu_j - b_j)y) \\ &= 0, \end{aligned}$$

a contradiction. Therefore $\rho \in \sigma(S)$.

If $\lambda \in P\sigma_r(a)$, $\mu \in P\sigma_r(b)$, say $x(\lambda_j - a_j) = y(\mu_j - b_j) = 0$ for some $x,y \in A \setminus \{0\}$, and $\rho \notin \sigma(S)$, we find $z,u \in A$ such that $xzy \neq 0$ and $(\rho - S)u = zy$. We now proceed as above and obtain again a contradiction. This accomplishes the proof. □

The following theorem has many forerunners. First of all there are the classical papers by Rosenblum [119] and Lumer and Rosenblum [102] in which (7.8.4) was obtained on $A = L(E)$ for $\delta_{a,b}$ and $R = \sum_{j=1}^{n} M_{f_j(a),g_j(b)}$, where f_j and g_j are holomorphic functions on $\sigma(a)$ and $\sigma(b)$, respectively.[2] Harte proved (7.8.1) and (7.8.2) with σ_ℓ and σ_r in place of $AP\sigma$ and $AP\sigma_r$ for general S and $A = L(H)$ ([72], thm 3.4). Weaker left-right inclusions than (7.8.1) (for $A = L(E)$) and (7.8.2) (for $A = L(H)$) were obtained by Davis and Rosenthal in

[2] The result on $\delta_{a,b}$ is attributed to D.C.Kleinecke.

[44], thm 3, and (7.8.1) through (7.8.3) were established by Fialkow for $A = L(H)$ ([60], thms 2.3 and 2.8). He also proved the same statements for symmetrically-normed ideals of $L(H)$. In [51] Embry and Rosenblum studied some particular cases of (7.8.4) on $L(E)$. Carrillo and Hernández proved (7.8.4) in the case $A = L(H)$ [25], while Fialkow ([61], p.417) deduced the same from results of Curto. Recently, Gravner [66] proved all identities for the Calkin algebra $C(H)$ by using Voiculescu's Theorem. The analogue of (7.8.4) with the Taylor spectrum in place of the Harte spectrum in the case $A = L(H)$ can be found in [37], thm 1; see also [40], cor. 6.1. From results of Halpern ([69], thm 1) and Smith [125] the identity (7.8.4) follows for $\delta_{a,b}$ and $M_{a,b}$ on arbitrary factors, and (7.8.1) through (7.8.3) for $M_{a,b}$ on a factor can be found in [108], thm 2.1. Finally, parts of 7.8 were proved for general S on primitive C*-algebras in [109].

Theorem 7.8. *Let* A *be an ultraprime unital Banach algebra, and for commuting n-tuples* $a,b \in A^n$ *let* $S = \sum_{j=1}^{n} M_{a_j,b_j}$ *be the corresponding elementary operator. Then*

$$(7.8.1) \qquad AP\sigma(S) = \{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in AP\sigma(a),\ \mu \in AP\sigma_r(b) \}.$$

If A *is a prime unital C*-algebra then also*

$$(7.8.2) \qquad AP\sigma_r(S) = \{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in AP\sigma_r(a),\ \mu \in AP\sigma(b) \}.$$

In this case

$$(7.8.3) \qquad AP\sigma(S) = \sigma_\ell(S) \quad \text{and} \quad AP\sigma_r(S) = \sigma_r(S),$$

as well as

$$(7.8.4) \qquad \sigma(S) = \{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in \sigma(a),\ \mu \in \sigma(b) \}.$$

Proof. ad (7.8.1) By Corollary 7.6 we only have to prove

$$APσ(S) \subseteq \{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in APσ(a),\ \mu \in APσ_r(b) \}.$$

But this holds for any unital Banach algebra since

$$APσ(L_a,R_b) \subseteq APσ(L_a) \times APσ(R_b)$$

$$= APσ(a) \times APσ_r(b)$$

by the Banach algebra version of A.2.7 (cf. [72], 3.1), so that we can apply the Spectral Mapping Theorem A.2.5 to the polynomial $p(\zeta_1,\ldots,\zeta_{2n}) = \sum_{j=1}^{n} \zeta_j \zeta_{j+n}$ (note that $S = p(L_a,R_b)$). For any unital algebra we also have

$$σ_\ell(L_a,R_b) \subseteq σ_\ell(a) \times σ_r(b)$$

and

$$σ_r(L_a,R_b) \subseteq σ_r(a) \times σ_\ell(b),$$

hence applying A.2.5 again we obtain

$$σ_\ell(S) \subseteq \{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in σ_\ell(a),\ \mu \in σ_r(b) \}$$

and

$$σ_r(S) \subseteq \{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in σ_r(a),\ \mu \in σ_\ell(b) \}.$$

Now if A is a prime unital C*-algebra we can use A.2.6 and Corollary 7.6 to obtain (7.8.2) and (7.8.3).

Finally we prove (7.8.4). As above $σ(L_a,R_b) \subseteq σ(a) \times σ(b)$, thus

$$σ(S) \subseteq \{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in σ(a),\ \mu \in σ(b) \}.$$

In order to simplify the notation we use $X \cdot Y$ as an abbreviation for the set $\{ \sum_{j=1}^{n} \alpha_j \beta_j \mid (\alpha_1,\ldots,\alpha_n) \in X,\ (\beta_1,\ldots,\beta_n) \in Y \}$ whenever $X, Y \subseteq \mathbb{C}^n$. Then

$$σ(a) \cdot σ(b) = [σ_\ell(a) \cup σ_r(a)] \cdot [σ_\ell(b) \cup σ_r(b)]$$

$= [AP\sigma(a) \cup AP\sigma_r(a)] \cdot [AP\sigma(b) \cup AP\sigma_r(b)]$ by A.2.6

$= AP\sigma(a) \cdot AP\sigma(b) \cup AP\sigma(a) \cdot AP\sigma_r(b)$

$\cup AP\sigma_r(a) \cdot AP\sigma(b) \cup AP\sigma_r(a) \cdot AP\sigma_r(b)$

$\subseteq \sigma(S)$ by 7.6 and 7.7

yields the final assertion. □

Corollary 7.9. *Let* A *be a prime unital* C^**-algebra and let* $a,b \in A^n$ *be commuting* n*-tuples. If* $S = \sum_{j=1}^{n} M_{a_j,b_j}$ *and* a,b *are normal (i.e. all* a_j *and* b_j *are normal), then*

$$\sigma(S) = AP\sigma(S) = AP\sigma_r(S).$$

Proof. If a is normal then $\sigma_\ell(a) = \sigma_r(a)$ by A.2.6, thus the assertion follows from 7.8. □

The next corollary was observed by Davis and Rosenthal ([44], thms 4 and 5) and by Fialkow ([53], thm 3.8) in the cases $A = L(H)$ and $A = C(H)$, respectively.

Corollary 7.10. *Let* A *be a prime unital* C^**-algebra and let* $a,b \in A$. *Then*

$$AP\sigma(\delta_{a,b}) = AP\sigma(a) - AP\sigma_r(b)$$

and

$$AP\sigma_r(\delta_{a,b}) = AP\sigma_r(a) - AP\sigma(b).$$

Proof. From Theorem 7.8 we conclude $AP\sigma(\delta_{a,b}) =$

$$\{\lambda_1\mu_1 + \lambda_2\mu_2 \mid (\lambda_1,\lambda_2) \in AP\sigma((a,1)),\ (\mu_1,\mu_2) \in AP\sigma_r((1,-b))\}$$

and $AP\sigma_r(\delta_{a,b}) =$

$$\{\lambda_1\mu_1 + \lambda_2\mu_2 \mid (\lambda_1,\lambda_2) \in AP\sigma_r((a,1)),\ (\mu_1,\mu_2) \in AP\sigma((1,-b))\}.$$

Since $AP\sigma((a,1)) = AP\sigma(a) \times \{1\}$ and $AP\sigma_r((a,1)) = AP\sigma_r(a) \times \{1\}$, we arrive at the assertion. □

In particular, a generalized inner derivation $\delta_{a,b}$ on a prime unital C*-algebra is surjective if and only if $AP\sigma_r(a) \cap AP\sigma(b) = \emptyset$, answering a question in [44].

We also derive the classical Lumer-Rosenblum result ([102], thm 10) in the setting of prime C*-algebras.

Corollary 7.11. *Let* A *be a prime unital C*-algebra and let* $a,b \in A$. *If* f_j, g_j *are holomorphic functions on* $\sigma(a)$, $\sigma(b)$ *respectively then the spectrum of* $R = \sum_{j=1}^{n} M_{f_j(a),g_j(b)}$ *is given by*

$$\sigma(R) = \{ \sum_{j=1}^{n} f_j(\alpha) g_j(\beta) \mid \alpha \in \sigma(a),\ \beta \in \sigma(b) \}.$$

Proof. By Theorem 7.8, $\sigma(R) =$

$$\{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in \sigma(f_1(a),\ldots,f_n(a)),\ \mu \in \sigma(g_1(b),\ldots,g_n(b)) \}$$

which by (A.2.5.3) equals the set

$$\{ \sum_{j=1}^{n} f_j(\alpha) g_j(\beta) \mid \alpha \in \sigma(a),\ \beta \in \sigma(b) \}. \square$$

Remark 7.12. The commutativity assumptions in Theorem 7.8 can be weakened. Proceeding along the same lines as in [72], thm 3.5 and [109], cor. 4.3 one obtains for example

$$\sigma_\ell(S) = \bigcup_{s \in \sigma_\ell(a)} \sigma_r(p(s,b))$$

where $S = \sum_{j=1}^{n} M_{a_j,b_j} \in E\ell(A)$ for a prime unital C*-algebra A, $a,b \in A^n$, a is a commuting n-tuple, and $p(\zeta_1,\ldots,\zeta_{2n}) = \sum_{j=1}^{n} \zeta_j \zeta_{j+n}$. Similary for the right spectrum.

The next result can be considered as some sharpening of (7.8.4). For $A = L(H)$ and a,b commuting n-tuples in A^n, a proof using the Taylor spectrum was given in [40], cor. 6.10 (it was first proved in this case in [72], thm 3.4).

Theorem 7.13. *Let* A *be a prime unital C*-algebra and let* $a \in A^n$, $b \in A^m$. *Then*

$$\sigma(L_a,R_b) = [\sigma_\ell(a) \times \sigma_r(b)] \cup [\sigma_r(a) \times \sigma_\ell(b)].$$

Proof. We have

$$\begin{aligned}
\sigma(L_a,R_b) &= \sigma_\ell(L_a,R_b) \cup \sigma_r(L_a,R_b) \\
&\subseteq [\sigma_\ell(a) \times \sigma_r(b)] \cup [\sigma_r(a) \times \sigma_\ell(b)] \\
&= [AP\sigma(a) \times AP\sigma_r(b)] \cup [AP\sigma_r(a) \times AP\sigma(b)] \text{ by A.2.6} \\
&\subseteq AP\sigma(L_a,R_b) \cup AP\sigma_r(L_a,R_b) \\
&\subseteq \sigma(L_a,R_b).
\end{aligned}$$

The last but one inclusion uses the primeness of A in the same way as Corollary 7.6. Compare [108], lem. 3.4 or [109], lem 2.10, and use the "twenty-one sets" lemma A.2.7. □

The main obstacle for a generalization of (7.8.2) through (7.8.4) to arbitrary ultraprime Banach algebras is the left-right asymmetry of the (joint) spectrum. For example, if T is a bounded surjective operator on a Banach space E which is not right invertible, then $0 \in AP\sigma_r(L_T) = \sigma_r(T)$, but 0 does not belong to $AP\sigma_r(T)$. For the remainder of this chapter we therefore concentrate on C*-algebras.

It can happen that in a homomorphic image of a C*-algebra approximate eigenvalues convert into proper eigenvalues, for example in any ultrapower (A.2.9), in the Berberian-Quigley extension ([118], 1.5.11), or in the Calkin algebra ([63], thm 4.1). Then, also the spectra of elementary operators coalesce. We prepare this with the following lemma.

Lemma 7.14. *Let* A *be a unital C*-algebra with the property that every left topological divisor of zero is a left divisor of zero. Then, for any* $a \in A^n$,

$$AP\sigma(a) = P\sigma(a) \quad \textit{and} \quad AP\sigma_r(a) = P\sigma_r(a).$$

Proof. If $\lambda \in AP\sigma(a)$ then, by A.2.6 and the hypothesis, there is a non-zero element $x \in A$ such that $\sum_{j=1}^{n} (\lambda_j - a_j)^*(\lambda_j - a_j)x = 0$. Therefore $\sum_{j=1}^{n} x^*(\lambda_j - a_j)^*(\lambda_j - a_j)x = 0$ which implies $(\lambda_j - a_j)x = 0$ for all $1 \leq j \leq n$. Hence $AP\sigma(a) = P\sigma(a)$. The second assertion follows by taking adjoints. □

Proposition 7.15. *Let* A *be a unital C*-algebra with the property that every left topological divisor of zero is a left divisor of zero. Let* $a,b \in A^n$ *be commuting n-tuples and let* $S = \sum_{j=1}^{n} M_{a_j,b_j}$ *be the corresponding elementary operator. Then*

(i) $$\sigma_\ell(S) \subseteq \{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in P\sigma(a),\ \mu \in P\sigma_r(b) \},$$

(ii) $$\sigma_r(S) \subseteq \{ \sum_{j=1}^{n} \lambda_j \mu_j \mid \lambda \in P\sigma_r(a),\ \mu \in P\sigma(b) \}.$$

If A *is prime then there is equality both in* (i) *and* (ii), *hence* $\sigma_\ell(S) = AP\sigma(S) = P\sigma(S)$ *and* $\sigma_r(S) = AP\sigma_r(S) = P\sigma_r(S)$.

Proof. We have

$$\begin{aligned} \sigma_\ell(L_a,R_b) &\subseteq \sigma_\ell(a) \times \sigma_r(b) \\ &= AP\sigma(a) \times AP\sigma_r(b) && \text{by A.2.6} \\ &= P\sigma(a) \times P\sigma_r(b) && \text{by 7.14.} \end{aligned}$$

Hence applying the Spectral Mapping Theorem (A.2.5.2) to the polynomial $p(\zeta_1,\ldots,\zeta_{2n}) = \sum_{j=1}^{n} \zeta_j \zeta_{j+n}$ we obtain (i). The assertion (ii) is proved in the same manner.

If A is a prime C*-algebra, by Proposition 7.5, the right hand side of (i) resp. (ii) is contained in $P\sigma(S)$ resp.

$P\sigma_r(S)$. This together with the obvious inclusions $P\sigma(S) \subseteq AP\sigma(S) \subseteq \sigma_\ell(S)$ and $P\sigma_r(S) \subseteq AP\sigma_r(S) \subseteq \sigma_r(S)$ completes the proof. □

The above equalities were observed in the case $A = \mathcal{C}(H)$ by Fialkow for $\delta_{a,b}$ in [53], thm 3.8 and [57], cor. 2.2, while Weber proved part of (ii) for $M_{a,b}$ ([145], lem.7), and Gravner [66] obtained (i) for general S. On an arbitrary prime C*-algebra the inclusions $P\sigma(S) \subseteq AP\sigma(S)$ and $P\sigma_r(S) \subseteq AP\sigma_r(S)$ can, of course, be strict. There are examples on $A = \mathcal{L}(H)$ which show that $\delta_{a,b}$ can be injective but not bounded below ([53], p.166), and that the range of $\delta_{a,b}$ can be proper but dense ([54], p.335). In particular, an extension of Proposition 7.4 to more general elementary operators fails.

If A is a W*-algebra, then one may ask for conditions which imply that the range of an elementary operator is $\sigma(A,A_*)$- dense. Our next result gives a complete answer to this question for multiplications. It was obtained in the special case $A = \mathcal{L}(H)$ by Weber ([145], thm 6).

Proposition 7.16. *Let* A *be a W*-algebra with predual* A_* *and let* $a,b \in A$. *The multiplication* $M_{a,b}$ *has* $\sigma(A,A_*)$*-dense range if and only if* a *is not a right divisor of zero and* b *is not a left divisor of zero.*

Proof. If $M_{a,b}A$ is $\sigma(A,A_*)$-dense in A, there exists a net (x_α) with $ax_\alpha b \to 1$ σ-weakly. For all $x,y \in A$ we have therefore $xax_\alpha b \to x$ and $ax_\alpha by \to y$ σ-weakly; hence a cannot be a right divisor of zero and b cannot be a left divisor of zero.

Suppose now that a is not a right divisor of zero and take $\phi \in A_*$ with $\langle \phi, ax \rangle = 0$ for all $x \in A$. The polar decomposi-

tion ([116], 3.6.7) of ϕ yields

$$<|\phi|, uaxa^*u^*> = <\phi, axa^*u^*> = 0$$

for all $x \in A$. Hence, the positive functional $|\phi|$ vanishes when it is restricted to $B = \overline{M_{ua,a^*u^*}A}$, which is a hereditary C*-subalgebra of A. If $B = 0$, then $ua = 0$ which implies that $u = 0$ and $\phi = 0$. If $B \neq 0$, then $|\phi|$ vanishes on A ([116], 3.1.6) which implies that $\phi = 0$, too. This shows that L_aA is $\sigma(A,A_*)$-dense in A. If, in addition, b is not a left divisor of zero, then b^* is not a right divisor of zero, and thus $R_bA = (L_{b^*}A)^*$ is $\sigma(A,A_*)$-dense in A. Now

$$A = \overline{L_aA}^{\sigma} = \overline{L_a\,\overline{R_bA}^{\sigma}}^{\sigma} \subseteq \overline{L_aR_bA}^{\sigma} = \overline{M_{a,b}A}^{\sigma}$$

completes the proof (with $\overline{}^{\sigma}$ denoting σ-weak closure). □

For an element x in a W*-algebra A, the left and the right support of x will be denoted by ℓ_x and r_x, respectively.

Corollary 7.17. *Let A be a W*-algebra and let $a,b \in A$. The following conditions are equivalent:*

(a) $M_{a,b}$ *has $\sigma(A,A_*)$-dense range.*

(b) $M_{b,a}$ *is injective.*

(c) $\ell_a = r_b = 1$.

Proof. There are the following descriptions of the null spaces of the left and right regular representation: $L_b^{-1}(0) = (1 - r_b)A$ and $R_a^{-1}(0) = A(1 - \ell_a)$. Hence, the equivalence of (a) and (c) follows from 7.16, while (b) $\Leftrightarrow$ (c) is immediate from $0 \in P\sigma(M_{b,a}) \Leftrightarrow 0 \in P\sigma_p(a) \cup P\sigma(b)$ (see p.93, Remark). □

Corollary 7.18. *If A is a von Neumann algebra on a Hilbert space H then a multiplication on A has $\sigma(A,A_*)$-dense range if and only if it has weakly dense range.*

Proof. Since the weak (operator) topology on A is weaker than the σ-weak topology, the "only if"-part is clear. To prove the "if"-part just reproduce the first part of the proof of 7.16 under the hypothesis that $M_{a,b}A$ is weakly dense. Again it follows that a is not a right divisor of zero and b is not a left divisor of zero. Thus, by 7.16, $M_{a,b}A$ is $\sigma(A,A_*)$-dense. □

This corollary was also obtained by Weber (ibid., thm 6) in case $A = L(H)$.

The loss of injectivity and surjectivity can be located precisely in the W*-case.

Proposition 7.19. *Let A be a W*-algebra and let $a,b \in A$. The multiplication $M_{a,b}$ maps the algebra $M_{r_a,\ell_b}A$ one-to-one on a $\sigma(A,A_*)$-dense subalgebra of $M_{\ell_a,r_b}A$.*

Proof. Since $M_{a,b}(r_a x \ell_b) = \ell_a(axb)r_b$ we have $M_{a,b}(M_{r_a,\ell_b}A) \subseteq M_{\ell_a,r_b}A$. If $axb = 0$ then $ax \in R_b^{-1}(0) = A(1 - \ell_b)$, hence $ax\ell_b = 0$. This implies $x\ell_b \in L_a^{-1}(0) = (1 - r_a)A$ from which $r_a x \ell_b = 0$ follows. Thus, the restriction of $M_{a,b}$ to $M_{r_a,\ell_b}A$ is injective. Finally, $\overline{M_{a,b}(M_{r_a,\ell_b}A)}^{\sigma} = \overline{M_{a,b}A}^{\sigma} = M_{\ell_a,r_b}A$ by the definition of left and right support. □

In our last result we again apply the primeness, now of a W*-algebra, to obtain a spectral inclusion which in some sense strengthens Proposition 7.5.

Theorem 7.20. *Let A be a factor and, for $a,b \in A^n$, let $S = \sum_{j=1}^{n} M_{a_j,b_j}$ be the corresponding elementary operator. If $\lambda \in P\sigma_r(a)$ and $\mu \in P\sigma(b)$, then the range of $\sum_{j=1}^{n} \lambda_j \mu_j - S$ is not $\sigma(A,A_*)$-dense in A.*

Proof. If $\lambda \in P\sigma_r(a)$ and $\mu \in P\sigma(b)$ then, by A.2.7, $p = \bigwedge_j (1 - \ell_{\lambda_j - a_j}) \neq 0$ and $q = \bigwedge_j (1 - r_{\mu_j - b_j}) \neq 0$. Since A is prime we can choose a normal state ϕ which does not vanish on $M_{p,q}A$. Define $\psi \in A_*$ by $\langle \psi, x \rangle = \langle \phi, pxq \rangle$.

Then, for $\rho = \sum_{j=1}^{n} \lambda_j \mu_j$,

$$\begin{aligned}
\langle \psi, (\rho - S)x \rangle &= \langle \psi, \sum_{j=1}^{n} (\lambda_j \mu_j - M_{a_j, b_j})x \rangle \\
&= \langle \psi, \sum_{j=1}^{n} (M_{\lambda_j - a_j, b_j} x + M_{\lambda_j, \mu_j - b_j} x) \rangle \\
&= \sum_{j=1}^{n} \langle \psi, (1 - p)(\lambda_j - a_j)xb_j + \lambda_j x(\mu_j - b_j)(1 - q) \rangle \\
&= 0
\end{aligned}$$

for all $x \in A$. Thus ψ vanishes on $(\rho - S)A$. □

Concluding remarks. 1. It seems likely that a generalization of several results on the essential spectrum and on Fredholm properties of S obtained in [40], [55], [58], [59], [61], [25] for the case $A = L(H)$ is possible to prime unital C*-algebras.

2. Extensions of some of the results obtained in this chapter to arbitrary C*-and W*-algebras are proposed in [109].

APPENDIX 1

BACKGROUND ON ULTRAPRODUCTS

In this first appendix several basic properties of ultraproducts are gathered from [41], [42], [77], [16] and [26] providing us with the prerequisites for Chapter 1. As we are working within the category of normed spaces we, of course, do not use the set-theoretic ultraproduct, but the well known appropriate modification.

Let I be a non-empty set and $(E_i)_{i\in I}$ a family of normed spaces. Denote by $\ell^\infty(I,E_i)$ the space of all bounded families $(x_i)_{i\in I}$ with $x_i \in E_i$. If $\mathcal{U}$ is an ultrafilter on I we put $n_{\mathcal{U}}$ the subspace of all $(x_i)_{i\in I} \in \ell^\infty(I,E_i)$ with the property that $\lim_{\mathcal{U}} \|x_i\| = 0$, i.e. $(\|x_i\|)_{i\in I}$ tends to zero along $\mathcal{U}$. The quotient $\ell^\infty(I,E_i)/n_{\mathcal{U}}$ is called *the ultraproduct* of the normed spaces E_i (*with respect to the ultrafilter* $\mathcal{U}$) and will be denoted by $(E_i)_{\mathcal{U}}$. For each element $(x_i)_{i\in I}$ in $\ell^\infty(I,E_i)$ the corresponding equivalence class in $(E_i)_{\mathcal{U}}$ is denoted by $(x_i)_{\mathcal{U}}$. By $\|(x_i)_{\mathcal{U}}\| := \lim_{\mathcal{U}} \|x_i\|$ a norm is defined on $(E_i)_{\mathcal{U}}$.

Since its inception the ultraproduct construction has proved to be quite useful in many branches of Banach space theory, see [77] for a survey. We will use it to obtain enlargements of a given space E in which topological properties of E can be described by algebraic properties of the enlargement. To this end take $I = \mathbb{N}$, $E_i = E$ for all i

where E is a given normed space, and an ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then $(E_i)_{\mathcal{U}}$ is called *the ultrapower of* E (with respect to $\mathcal{U}$) and is denoted by $\hat{E}_{\mathcal{U}}$. The elements of $\hat{E}_{\mathcal{U}}$ will usually be denoted by $\hat{x}$, $\hat{y}$,..., and the space $n_{\mathcal{U}}$ will be specified as $n_{\mathcal{U}}(E)$ if necessary. There is a canonical isometric embedding J of E into $\hat{E}_{\mathcal{U}}$ defined by $Jx := (x,x,\ldots)_{\mathcal{U}}$ for all $x \in E$. Usually we suppress J or, by abuse of notation, write $\hat{x}$ for Jx.

Since we are only interested in proper enlargements of E, we will always assume that $\mathcal{U}$ is a *free* ultrafilter on $\mathbb{N}$, i.e. $\mathcal{U}$ does not contain any finite set. The first basic property of $\hat{E}_{\mathcal{U}}$ now is

A.1.1. The ultrapower $\hat{E}_{\mathcal{U}}$ of E with respect to a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ endowed with coordinatewise operations and norm as defined above has the following properties:

$\hat{E}_{\mathcal{U}}$ *is a Banach space if* E *is a normed space.*

$\hat{E}_{\mathcal{U}}$ *is a Banach algebra if* E *is a normed algebra.*

$\hat{E}_{\mathcal{U}}$ *is a C*-algebra if* E *is a C*-algebra.*

([77], prop. 3.1; [78], thm 1.3, see also [103], thm 3.16.1)

The next lemma is used in the proof of Lemma 1.1. It is a consequence of the fundamental property of an ultrafilter $\mathcal{U}$ that for each subset X of I either X or its complement belong to $\mathcal{U}$.[1]

A.1.2. *If* $\hat{x} \in \hat{E}_{\mathcal{U}}$ *has norm one then there is a representing sequence* $(x_n)_{n\in\mathbb{N}}$ *in* E *(i.e.* $\hat{x} = (x_n)_{\mathcal{U}}$*) such that* $\|x_n\| = 1$ *for all* $n \in \mathbb{N}$.

---

[1] One could also define ultrafilters as those filters $\mathcal{F}$ which are *prime* in the sense that $X \cup Y \in \mathcal{F}$ implies either $X \in \mathcal{F}$ or $Y \in \mathcal{F}$.

Proof. Let $N = \{n \in \mathbb{N} \mid x_n' = 0\}$ where $(x_n')_{n\in\mathbb{N}}$ is some representing sequence of $\hat{x}$. If $N \in U$ then the set U_k of all $n \in \mathbb{N}$ such that $\|x_n'\| < \frac{1}{k}$ would belong to U for all $k \in \mathbb{N}$ contradicting $\lim_U \|x_n'\| = 1$. Therefore $\mathbb{N} \setminus N \in U$. Take $m \in \mathbb{N} \setminus N$ and put $x_n = \frac{x_m'}{\|x_m'\|}$ for all $n \in N$ and $x_n = \frac{x_n'}{\|x_n'\|}$ for all $n \in \mathbb{N} \setminus N$. It follows that

$$\lim_U \|x_n' - x_n\| = \lim_U \frac{\| x_n'\|x_n'\| - x_n'\|}{\|x_n'\|} = \lim_U |\|x_n'\| - 1| = 0.$$

Thus $(x_n)_{n\in\mathbb{N}}$ is a representing sequence of unit vectors for $\hat{x}$.□

Lemma A.1.2 fails for non-ultrafilters which is the reason why we have to work with ultrapowers instead of reduced powers with respect to some filter finer than the Fréchet filter on $\mathbb{N}$. Clearly, in all of the above, $\mathbb{N}$ can be replaced by any denumerable set.

There is also a notion of 'product' of ultrafilters. For any set I let $P(I) = \{X \mid X \subseteq I\}$ denote its *power set*.

A.1.3. *Let U and V be free ultrafilters on $\mathbb{N}$. Then $U \times V := \{Y \in P(\mathbb{N} \times \mathbb{N}) \mid \{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in Y\} \in U\} \in V\}$ defines a free ultrafilter on $\mathbb{N} \times \mathbb{N}$.*

([26], exer. 6.5.1; [16], exer. 6.2.6)

Proof. If $\emptyset \in U \times V$ then $V \ni \{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in \emptyset\} \in U\} = \emptyset$, a contradiction. Obviously $\mathbb{N} \times \mathbb{N} \in U \times V$. If $Y_1, Y_2 \in U \times V$ then

$$\{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in Y_1 \cap Y_2\} \in U\} =$$

$$\{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in Y_1\} \cap \{i \in \mathbb{N} \mid (i,j) \in Y_2\} \in U\} =$$

$\{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in Y_1\} \in U\} \cap \{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in Y_2\} \in U\}$ belongs to V, thus $Y_1 \cap Y_2 \in U \times V$.

It is also easy to see that $Y_2 \in P(\mathbb{N} \times \mathbb{N})$ belongs to $U \times V$ whenever Y_2 contains a subset $Y_1 \in U \times V$, so that $U \times V$ is a filter on $\mathbb{N} \times \mathbb{N}$. Suppose now that $Y \in P(\mathbb{N} \times \mathbb{N})$ and that $Y^c := \mathbb{N} \times \mathbb{N} \setminus Y \notin U \times V$. Then $\{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in Y^c\} \in U\}$ is not in V. Thus, we obtain in succession

$$\{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in Y^c\} \in U\}^c \in V,$$

$$\{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in Y^c\} \notin U\} \in V,$$

$$\{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in Y^c\}^c \in U\} \in V,$$

$$\{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in Y\} \in U\} \in V$$

(where we used the assumption that V and U are ultrafilters in the first and the third line, respectively). Hence $Y \in U \times V$ which shows that $U \times V$ is an ultrafilter. Finally, in order to realize that $U \times V$ is free, suppose on the contrary that $\{(i_o,j_o)\} \in U \times V$ for some $i_o, j_o \in \mathbb{N}$. Then

$$V \ni \{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in \{(i_o,j_o)\}\} \in U\}$$
$$= \{j \in \mathbb{N} \mid \{i_o\} \in U,\ j = j_o\} = \emptyset ,$$

a contradiction.□

Finite iteration of ultrapowers is easily understood with the aid of products of ultrafilters.

A.1.4. *Let U and V be free ultrafilters on $\mathbb{N}$, and let A be a normed algebra. Then there is a natural isometric algebra isomorphism Φ from $(\hat{A}_U)\hat{}_V$ onto $\hat{A}_{U\times V}$. If A is a *-algebra then Φ is a *-homomorphism.*

(An analogous result for the set-theoretic ultrapower due to Frayne, Morel, and Scott can be found in [26], 6.5.2 and [16], exer.6.2.7.)

Proof. For a normed space E denote by $\ell^\infty(E)$ the space of bounded sequences in E. If $0 \neq \hat{x} \in (\hat{A}_U)\hat{}_V$, then take from $\hat{x}$ a representing sequence $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\hat{A}_U)$. By Lemma A.1.2 we may assume that $\|x_n\| = \|\hat{x}\|$ for each $n \in \mathbb{N}$. Choose from $x_n \in \hat{A}_U$ a representing sequence $(x_{nm})_{m\in\mathbb{N}} \in \ell^\infty(A)$ with $\|x_{nm}\| = \|\hat{x}\|$ for all $n,m \in \mathbb{N}$. If $\hat{x} = 0$, then every representing sequence $(x_n)_{n\in\mathbb{N}}$ of $\hat{x}$ belongs to n_V, i.e.

$$V_k = \{n \in \mathbb{N} \mid \|x_n\| < \tfrac{1}{k}\} \in V \quad \text{for all } k \in \mathbb{N}.$$

If $(x_{nm})_{m\in\mathbb{N}} \in x_n$ then

$$\{m \in \mathbb{N} \mid |\|x_{nm}\| - \|x_n\|| < \tfrac{1}{k}\} \in U \quad \text{for each } k \in \mathbb{N}.$$

Thus, for all $k \in \mathbb{N}$, $n \in V_k$, $U_{kn} = \{m \in \mathbb{N} \mid \|x_{nm}\| < \frac{2}{k}\} \in U$ and therefore $\{n \in \mathbb{N} \mid U_{kn} \in U\} \in V$. We now put

$$Y_k = \{(n,m) \in \mathbb{N}\times\mathbb{N} \mid \|x_{nm}\| < \tfrac{2}{k}\} \quad \text{and conclude}$$

$$\{j \in \mathbb{N} \mid \{i \in \mathbb{N} \mid (i,j) \in Y_k\} \in U\} \in V \quad \text{for each } k \in \mathbb{N},$$

which implies $(x_{nm})_{(n,m)\in\mathbb{N}\times\mathbb{N}} \in n_{U\times V}$.

It is now easy to verify that

$$\Phi : (\hat{A}_U)\hat{}_V \longrightarrow \hat{A}_{U\times V}, \quad \Phi(\hat{x}) := (x_{nm})_{U\times V}$$

defines an algebra isomorphism into $\hat{A}_{U\times V}$ which is also an isometry. If $\hat{z} \in \hat{A}_{U\times V}$, $\hat{z} \neq 0$ then we choose $(z_{nm})_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ with $\|z_{nm}\| = \|\hat{z}\|$ for all n,m (A.1.2) and put $\hat{x} = (z_n)_V \in (\hat{A}_U)\hat{}_V$, where $z_n := (z_{nm})_U$ for each $n \in \mathbb{N}$. Clearly $\Phi(\hat{x}) = \hat{z}$, hence Φ is surjective.

Finally Φ commutes with the involution if A is an involutive algebra.□

As a consequence we recover the iterated ultrapower of A with respect to free ultrafilters on $\mathbb{N}$ as an ultrapower of A with respect to another free ultrafilter on $\mathbb{N}$. This is used in Proposition 1.7.

A.1.5. *Let A be a normed (*-)algebra. If U and V are free ultrafilters on $\mathbb{N}$ then there exists a free ultrafilter W on $\mathbb{N}$ such that $(\hat{A}_U)\hat{}_V$ and $\hat{A}_W$ are isometrically isomorphic as Banach (*-)algebras.*

Proof. Let σ be a bijective mapping from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$. Then $W := \{X \subseteq \mathbb{N} \mid \sigma^{-1}(X) \in U \times V\}$ is an ultrafilter on $\mathbb{N}$ ([26], exer. 4.3.7). If $X_o \subseteq X$ for all $X \in W$ then $\sigma^{-1}(X_o) \subseteq \sigma^{-1}(X) \in U \times V$. Since σ is bijective, it follows that $\sigma^{-1}(X_o) \subseteq Y$ for all $Y \in U \times V$, hence $\sigma^{-1}(X_o) = \emptyset$ since $U \times V$ is free. Therefore, W is free, too.

For a sequence $(x_{nm})_{(n,m)\in\mathbb{N}\times\mathbb{N}} \in \ell^\infty(\mathbb{N}\times\mathbb{N}, A)$ we now define $\Psi((x_{nm})_{U\times V}) := (x_{\sigma(n,m)})_W$ where $x_{\sigma(n,m)} := x_{nm}$. In case $(x_{nm})_{(n,m)\in\mathbb{N}\times\mathbb{N}} \in n_{U\times V}$ then, for each $k \in \mathbb{N}$, the set $\{(n,m) \in \mathbb{N} \times \mathbb{N} \mid \|x_{nm}\| < \frac{1}{k}\}$ belongs to $U \times V$. Therefore,

$$\{j \in \mathbb{N} \mid j = \sigma(n,m),\ \|x_j\| < \tfrac{1}{k}\} \in W$$

which implies $(x_{\sigma(n,m)}) \in n_W$.

It now follows easily that Ψ is a (*-)algebra isomorphism from $\hat{A}_{U\times V}$ onto $\hat{A}_W$, and Ψ is clearly isometric (use A.1.2 again). The desired isomorphism from $(\hat{A}_U)\hat{}_V$ onto $\hat{A}_W$ is the composition of Ψ with Φ from A.1.4. □

In the following as well as in Appendix 2 we will also need ultraproducts of bounded linear mappings between normed spaces.

A.1.6. Let E and F be complex normed spaces and let $T \in L(E,F)$. If $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$ then $(Tx_n)_{n\in\mathbb{N}} \in n_{\mathcal{U}}(F)$ for every sequence $(x_n)_{n\in\mathbb{N}} \in n_{\mathcal{U}}(E)$. Thus, by $\hat{T}((x_n)_{\mathcal{U}}) := (Tx_n)_{\mathcal{U}}$ for each sequence $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(E)$ a bounded linear mapping from $\hat{E}_{\mathcal{U}}$ into $\hat{F}_{\mathcal{U}}$ is defined. It is easy to see that $T \mapsto \hat{T}$ is an isometric embedding of $L(E,F)$ into $L(\hat{E}_{\mathcal{U}},\hat{F}_{\mathcal{U}})$. If $E = F$ is a normed (*-)algebra, this embedding is an algebra homomorphism which maps (*-)algebra homomorphisms onto (*-)algebra homomorphisms. We have in addition

T is an isometry if and only if $\hat{T}$ is an isometry, and $\hat{T}$ is open if T is open

(compare A.2.8).

We now consider linear functionals on a normed space E and relate the ultrapower of the dual of E to the dual of the ultrapower of E.

A.1.7. Let E be a complex normed space with (topological) dual E' (this will not lead to confusion with the notation A' for the commutant of an algebra A). Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. There is an isometric embedding j of the ultrapower $(E')\hat{}_{\mathcal{U}}$ of E' into the dual $\hat{E}'_{\mathcal{U}}$ of the ultrapower $\hat{E}_{\mathcal{U}}$ of E given by $\langle j((\phi_n)_{\mathcal{U}}),(x_n)_{\mathcal{U}}\rangle := \lim_{\mathcal{U}} \langle \phi_n, x_n\rangle$ for every $(\phi_n)_{n\in\mathbb{N}} \in \ell^\infty(E')$ and $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(E)$ ([77], p.87; see also [79], sect. 8). The embedding j is not surjective unless $E_{\mathcal{U}}$ is reflexive ([77], prop. 7.1). If $T \in L(E,F)$ for some complex normed space F and $T' \in L(F',E')$ is its adjoint, we can regard $\widehat{T'} \in L((F')\hat{}_{\mathcal{U}},(E')\hat{}_{\mathcal{U}})$ as the restriction of the adjoint $\hat{T}' \in L(\hat{F}'_{\mathcal{U}},\hat{E}'_{\mathcal{U}})$ of $\hat{T}$ to the space $j((F')\hat{}_{\mathcal{U}})$ by means of the formula $\hat{T}'j((\phi_n)_{\mathcal{U}}) = j((T'\phi_n)_{\mathcal{U}})$ for every sequence $(\phi_n)_{n\in\mathbb{N}} \in \ell^\infty(F')$. This will be used in Lemma A.2.8.

As another application of A.1.6 we study ultrapowers of ideals and quotients of normed algebras. First we need an auxiliary result.

A.1.8. *If* E *and* F *are normed spaces and* $\mathcal{U}$ *is a free ultrafilter on* $\mathbb{N}$ *then* $(E \oplus F)\hat{}_{\mathcal{U}}$ *is isometrically isomorphic to* $\hat{E}_{\mathcal{U}} \oplus \hat{F}_{\mathcal{U}}$.

Proof. Define $\sigma: (E \oplus F)\hat{}_{\mathcal{U}} \to \hat{E}_{\mathcal{U}} \oplus \hat{F}_{\mathcal{U}}$ by $\sigma((x_n,y_n)_{\mathcal{U}}) := ((x_n)_{\mathcal{U}},(y_n)_{\mathcal{U}})$ for every sequence $(x_n,y_n)_{n\in\mathbb{N}} \in \ell^\infty(E \oplus F) = \ell^\infty(E) \oplus \ell^\infty(F)$. Since $n_{\mathcal{U}}(E \oplus F) = n_{\mathcal{U}}(E) \oplus n_{\mathcal{U}}(F)$, σ is a well-defined linear isomorphism onto $\hat{E}_{\mathcal{U}} \oplus \hat{F}_{\mathcal{U}}$. If both $E \oplus F$ and $\hat{E}_{\mathcal{U}} \oplus \hat{F}_{\mathcal{U}}$ are endowed with the ℓ^1-norm, then σ is also an isometry.□

A.1.9. *Let* A *be a (unital) normed algebra and* M *be a (unit linked)* A*-module. Let* $\mathcal{U}$ *be a free ultrafilter on* $\mathbb{N}$. *Then* $\hat{M}_{\mathcal{U}}$ *is a (unit linked) Banach* $\hat{A}_{\mathcal{U}}$*-module. In particular, if* I *is an ideal of* A *then* $\hat{I}_{\mathcal{U}}$ *is a closed ideal of* $\hat{A}_{\mathcal{U}}$.

Proof. We give the proof for left A-modules while the result for right modules and bimodules follows analogously. If M is a normed left A-module then $\hat{M}_{\mathcal{U}}$ is a Banach space by A.1.1. Let $T : A \times M \to M$, $T(a,m) = am$ denote the module multiplication on M. Then, by A.1.8, $(A \times M)\hat{}_{\mathcal{U}} = \hat{A}_{\mathcal{U}} \times \hat{M}_{\mathcal{U}}$, and thus $\hat{T} : (A \times M)\hat{}_{\mathcal{U}} \to \hat{M}_{\mathcal{U}}$ given by $\hat{T}((a_n,m_n)_{\mathcal{U}}) := (T(a_n,m_n))_{\mathcal{U}} = (a_nm_n)_{\mathcal{U}}$ for each sequence $(a_n,m_n)_{n\in\mathbb{N}} \in \ell^\infty(A \times M)$ defines a module multiplication $((a_n)_{\mathcal{U}},(m_n)_{\mathcal{U}}) \mapsto (a_nm_n)_{\mathcal{U}}$ on $\hat{M}_{\mathcal{U}}$ (use A.1.6). Since $\|T\| = \|\hat{T}\|$, the mapping $(\hat{a},\hat{m}) \to \hat{a}\hat{m}$ is also bounded; thus $\hat{M}_{\mathcal{U}}$ is a Banach left $\hat{A}_{\mathcal{U}}$-module. Clearly, if A is unital and M is unit linked then $\hat{A}_{\mathcal{U}}$ is unital and $\hat{M}_{\mathcal{U}}$ is unit linked.

Since the ideals of A are the sub-bimodules of the A-bimodule A, $\hat{I}_u$ is an ideal of $\hat{A}_u$ whenever I is an ideal of A. Since $\hat{I}_u$ is complete, it is closed in $\hat{A}_u$.□

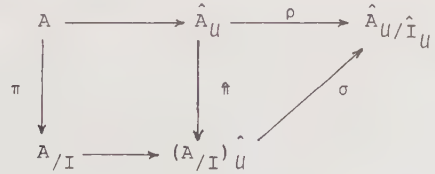
A.1.10. *Let I be a closed ideal of the normed algebra A, and let u be a free ultrafilter on $\mathbb{N}$. Then the ultrapower $(A_{/I})\hat{}_u$ of the quotient algebra $A_{/I}$ and the quotient algebra $\hat{A}_u/\hat{I}_u$ are isometrically isomorphic.*

Proof. We give the details of the argument indicated in [41], prop. 4. Let $\pi: A \to A_{/I}$ be the canonical homomorphism onto $A_{/I}$. Since π is open, $\Uparrow$ is an open homomorphism from $\hat{A}_u$ onto $(A_{/I})\hat{}_u$ by A.1.6. If $\hat{x} = (x_n)_u \in \hat{A}_u$, by definition of the quotient norm, there is a bounded sequence $(z_n)_{n\in\mathbb{N}}$ in I such that $\|x_n + z_n\| \leq \|\pi(x_n)\| + \frac{1}{n}$ for each $n \in \mathbb{N}$. Therefore,

$$\|\hat{x} + \hat{I}_u\| = \inf\{\lim_u \|x_n + y_n\| \mid (y_n)_{n\in\mathbb{N}} \in \ell^\infty(I)\}$$

$$\leq \lim_u \|x_n + z_n\| \leq \lim_u (\|\pi(x_n)\| + \tfrac{1}{n}) = \|\Uparrow(\hat{x})\|,$$

i.e. (1) $\|\rho(\hat{x})\| \leq \|\Uparrow(\hat{x})\|$ where $\rho: \hat{A}_u \to \hat{A}_u/\hat{I}_u$ is the canonical homomorphism onto $\hat{A}_u/\hat{I}_u$. In particular, it follows from (1) that $\ker \Uparrow \subseteq \ker \rho$. Thus the mapping $\sigma: (A_{/I})\hat{}_u \to \hat{A}_u/\hat{I}_u$ is well-defined by $\sigma \circ \Uparrow = \rho$, and σ is clearly a surjective algebra homomorphism.



By (1), $\|\sigma(\hat{\Phi})\| \leq \|\hat{\Phi}\|$ for each $\hat{\Phi} \in (A_{/I})^{\hat{}}_U$. To prove the converse inequality let $\hat{\Phi} = \hat{\pi}(\hat{x})$ for some $\hat{x} = (x_n)_U \in \hat{A}_U$ and observe that $\|\hat{\pi}(\hat{x})\| = \lim_U \|\pi(x_n)\| \leq \lim_U \|x_n + y_n\|$ for every bounded sequence $(y_n)_{n\in\mathbb{N}}$ in I. Therefore, $\|\hat{\pi}(\hat{x})\| \leq \inf\{\lim_U \|x_n + y_n\| \mid (y_n)_{n\in\mathbb{N}} \in \ell^\infty(I)\} = \|\hat{x} + \hat{I}_U\| = \|\rho(\hat{x})\|$ shows that $\|\hat{\Phi}\| \leq \|\sigma(\hat{\Phi})\|$.
By means of this, σ is the desired isometric isomorphism.□

Remark. If I is a not necessarily closed ideal of A, almost the same proof shows that $(A_{/\bar{I}})^{\hat{}}_U$ and $\hat{A}_{U/\hat{I}_U}$ are isometrically isomorphic.

We conclude with describing the ultrapowers of two other standard constructions that yield new normed algebras from given ones.

A.1.11. For a complex normed algebra A without unit denote by $A + \mathbb{C}$ its *unitization*. Let U be a free ultrafilter on $\mathbb{N}$. Then we have

The algebras $(A + \mathbb{C})^{\hat{}}_U$ *and* $\hat{A}_U + \mathbb{C}$ *are isometrically isomorphic.*

Proof. Since $\hat{\mathbb{C}}_U = \mathbb{C}$, it follows from A.1.8 that $(A + \mathbb{C})^{\hat{}}_U = \hat{A}_U + \mathbb{C}$ (as Banach spaces). It is easy to verify that this identification also preserves the multiplication on the unitization.□

A.1.12. Let A be a unital normed algebra. For each $n \in \mathbb{N}$ let $M_n(A)$ denote the algebra of $n\times n$-matrices with entries from A. We consider $M_n(A)$ as a subalgebra of $L(A^n)$, where A^n is endowed with the ℓ^1-norm. Then the operator norm of $(a(i,j))_{1\leq i,j\leq n} \in M_n(A)$ is given by $\|(a(i,j))_{1\leq i,j\leq n}\| =$

$\max_j \sum_{i=1}^{n} \|a(i,j)\|$. If $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$ we have

$M_n(A)\hat{}_{\mathcal{U}}$ and $M_n(\hat{A}_{\mathcal{U}})$ are isometrically isomorphic Banach algebras.

Proof. By A.1.1, $M_n(A)\hat{}_{\mathcal{U}}$ and $M_n(\hat{A}_{\mathcal{U}})$ are Banach algebras. Define $\sigma: M_n(A)\hat{}_{\mathcal{U}} \to M_n(\hat{A}_{\mathcal{U}})$ by $\sigma((a_k)_{\mathcal{U}}) := ((a_k(i,j))_{\mathcal{U}})_{1\le i,j\le n}$ for each sequence $(a_k)_{k\in\mathbb{N}} \in \ell^\infty(M_n(A))$, $a_k = (a_k(i,j))_{1\le i,j\le n}$. Obviously, σ is a well-defined surjective algebra homomorphism. From

$$\begin{aligned}\|\sigma((a_k)_{\mathcal{U}})\| &= \max_j \sum_{i=1}^{n} \lim_{\mathcal{U}} \|a_k(i,j)\| \\ &= \lim_{\mathcal{U}} \max_j \sum_{i=1}^{n} \|a_k(i,j)\| \\ &= \lim_{\mathcal{U}} \|a_k\| = \|(a_k)_{\mathcal{U}}\|\end{aligned}$$

we see that σ is an isometry.□

APPENDIX 2

BACKGROUND ON JOINT SPECTRA

In this appendix we shall collect all prerequisites on joint spectra in the sense of R.E.Harte [71] that are necessary in Chapter 7. Although we have not been able to find references for a few, most of these results are well known but scattered through the literature. Thus it seems convenient to recall them at this place.

Throughout A will denote a unital complex Banach algebra and 1 will denote its identity. We start with the definition of joint spectra for an n-tuple $a \in A^n$ (and usually omit the words 'joint' and 'left', but not 'right').

A.2.1. If $a = (a_1,\dots,a_n) \in A^n$ is an n-tuple of elements of A we call

$$P\sigma(a) := \{\lambda \in \mathbb{C}^n \mid \exists\, 0 \neq x \in A \ \forall\, 1 \leq j \leq n\colon (\lambda_j - a_j)x = 0\}$$

the point spectrum of a,

$$P\sigma_r(a) := \{\lambda \in \mathbb{C}^n \mid \exists\, 0 \neq x \in A \ \forall\, 1 \leq j \leq n\colon x(\lambda_j - a_j) = 0\}$$

the right point spectrum of a,

$$AP\sigma(a) := \{\lambda \in \mathbb{C}^n \mid \inf_{\substack{x\in A\\ \|x\|=1}} \sum_{j=1}^{n} \|(\lambda_j - a_j)x\| = 0\}$$

the approximate point spectrum of a,

$$AP\sigma_r(a) := \{\lambda \in \mathbb{C}^n \mid \inf_{\substack{x\in A\\ \|x\|=1}} \sum_{j=1}^{n} \|x(\lambda_j - a_j)\| = 0\}$$

the right approximate point spectrum of a,

$\sigma_\ell(a) := \{\lambda \in \mathbb{C}^n \mid 1 \notin \sum_{j=1}^{n} A(\lambda_j - a_j)\}$ *the left spectrum of* a,

$\sigma_r(a) := \{\lambda \in \mathbb{C}^n \mid 1 \notin \sum_{j=1}^{n} (\lambda_j - a_j)A\}$ *the right spectrum of* a,

and $\sigma(a) := \sigma_\ell(a) \cup \sigma_r(a)$ *the spectrum of* a.

If $0 = (0,\ldots,0) \notin \sigma_\ell(a)$ we say that $a \in A^n$ is *left invertible in* A, analogously we define *right invertible* and *invertible*.

These spectra are natural generalizations from the case n = 1. With exception of the point spectra, they are all compact subsets of $\mathbb{C}^n$ and non-void, if a is a *commuting* n-tuple, i.e. all the a_j mutually commute. If $A = L(E)$ and n = 1 we have ([71], thm 2.4, [17], 57.6 and 57.17)

$\lambda \in P\sigma(T)$ *iff* $\lambda - T$ *is not injective*,

$\lambda \in P\sigma_r(T)$ *iff* $\lambda - T$ *does not have dense range*,

$\lambda \in AP\sigma(T)$ *iff* $\lambda - T$ *is not bounded below*,

$\lambda \in AP\sigma_r(T)$ *iff* $\lambda - T$ *is not surjective*,

whence $P\sigma_r(T)$ and $AP\sigma_r(T)$ are also referred to as the *compression spectrum* and the *defect spectrum* of T, resp.

There is a similar description for arbitrary n-tuples of operators as well as a 'duality theory'.

A.2.2. If E is a non-zero complex Banach space and $A = L(E)$ we denote by $T' = (T_1',\ldots,T_n')$ the n-tuple of adjoints of an n-tuple $T = (T_1,\ldots,T_n)$ of operators on E. Then we have ([71], thm 2.4)

$$P\sigma(T) = \{\lambda \in \mathbb{C}^n \mid \bigcap_{j=1}^{n} \ker(\lambda_j - T_j) \neq \{0\}\} \subseteq P\sigma_r(T'),$$

$$P\sigma_r(T) = \{\lambda \in \mathbb{C}^n \mid \sum_{j=1}^{n} (\lambda_j - T_j)E \text{ is not dense in } E\} = P\sigma(T'),$$

$$AP\sigma(T) = \{\lambda \in \mathbb{C}^n \mid \inf_{\substack{x\in E \\ \|x\|=1}} \sum_{j=1}^{n} \|(\lambda_j - T_j)x\| = 0\} = AP\sigma_r(T')$$

and

$$AP\sigma_r(T) = \{\lambda \in \mathbb{C}^n \mid \sum_{j=1}^{n} (\lambda_j - T_j)E \neq E\} = AP\sigma(T').$$

A.2.3. If A is a commutative algebra with spectrum $\hat{A}$ then

$$\sigma(a) = \{ (\phi(a_1),\ldots,\phi(a_n)) \mid \phi \in \hat{A}\}.$$

This is immediate from Gelfand theory.

The following fact is needed in the proof of Proposition 7.1.

A.2.4. For every unital ring R denote by $\mathrm{Inv}(R)$ the group of invertible elements of R. For $a,b \in R$ we have

$$\{a,b\} \subseteq \mathrm{Inv}(R) \text{ *if and only if* } \{ab,ba\} \subseteq \mathrm{Inv}(R).$$

Proof. The "only if"-part is clear. Conversely, $ab(ab)^{-1} = 1$ implies that a is right invertible, and $(ba)^{-1}ba = 1$ implies that a is left invertible. Similarly for b.□

The joint spectra defined in A.2.1 are subject to some Spectral Mapping Theorems.

A.2.5. *Let* σ^* *be any of the spectra in* A.2.1. *If* $a \in A^n$ *is an arbitrary* n*-tuple and* $p = (p_1,\ldots,p_r)$ *is a system of polynomials* p_k *each of them in* n *variables, then*

(A.2.5.1) $$p(\sigma^*(a)) \subseteq \sigma^*(p(a)).$$

If a *is a commuting* n*-tuple then, except for the point spectra, the converse inclusions also hold*

(A.2.5.2) $$\sigma^*(p(a)) \subseteq p(\sigma^*(a)).$$

If a *is a commuting* n*-tuple, if* $\sigma^* = \sigma$, σ_ℓ *or* σ_r, *and*

if $f:U \to \mathbb{C}^r$ *is a system of functions holomorphic on an open set* $U \subseteq \mathbb{C}^n$ *containing* $\sigma_B(a)$ *for some closed commutative subalgebra* $B \subseteq A$ *with* $1 \in B$ *then*

$$f(\sigma_A^*(a)) = \sigma_A^*(f(a)). \tag{A.2.5.3}$$

The results in (A.2.5.1) and (A.2.5.3) can be found in [71], thm 3.2 and [73], thm 1, respectively, while (A.2.5.2) is a combination of [71], thm 4.3 and [31], thm 1.

Our next result which is taken from [109], lem. 2.3 corresponds to the fact that a bounded operator T on a Hilbert space H is bounded below if and only if T is left invertible in $L(H)$, and is surjective if and only if T is right invertible in $L(H)$ ([17], 57.19 and 57.20). Because of its importance for the description of the spectra of elementary operators (see Thms 7.8 and 7.13), we include its proof.

A.2.6. *Let* A *be a unital* C^**-algebra and* $a \in A^n$. *Then*

(i) $\sigma_\ell(a) = AP\sigma(a)$

$$= \{\lambda \in \mathbb{C}^n \mid \sum_{j=1}^n (\lambda_j - a_j)^*(\lambda_j - a_j) \text{ is singular in } A\},$$

(ii) $\sigma_r(a) = AP\sigma_r(a)$

$$= \{\lambda \in \mathbb{C}^n \mid \sum_{j=1}^n (\lambda_j - a_j)(\lambda_j - a_j)^* \text{ is singular in } A\},$$

(iii) *If* a *is normal (i.e. if all* a_j *are normal) then*

$$\sigma(a) = AP\sigma(a) = AP\sigma_r(a).$$

Proof. The inclusion $AP\sigma(a) \subseteq \sigma_\ell(a)$ is always true. Next $\sigma_\ell(a) \subseteq \{\lambda \in \mathbb{C}^n \mid \sum_{j=1}^n (\lambda_j - a_j)^*(\lambda_j - a_j)$ is singular in $A\}$ holds since an inverse b of $\sum_{j=1}^n (\lambda_j - a_j)^*(\lambda_j - a_j)$ yields a left inverse $(b(\lambda_1 - a_1)^*, \ldots, b(\lambda_n - a_n)^*)$ of $(\lambda_1 - a_1, \ldots, \lambda_n - a_n)$.

Now, if $\lambda \in \mathbb{C}^n$ such that $\sum_{j=1}^{n} (\lambda_j - a_j)^*(\lambda_j - a_j)$ is singular in A then there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in the C*-subalgebra B generated by $\sum_{j=1}^{n} (\lambda_j - a_j)^*(\lambda_j - a_j)$ and 1 such that $\|x_k\| = 1$ for all k and $\lim_{k\to\infty} \| \sum_{j=1}^{n} (\lambda_j - a_j)^*(\lambda_j - a_j)x_k\| = 0$ which is immediate by the functional calculus. Hence,

$$\| \sum_{j=1}^{n} x_k^*(\lambda_j - a_j)^*(\lambda_j - a_j)x_k\| \geq \max_{1\leq j\leq n} \|x_k^*(\lambda_j - a_j)^*(\lambda_j - a_j)x_k\|$$

yields $\lim_{k\to\infty} \sum_{j=1}^{n} \|(\lambda_j - a_j)x_k\| = 0$, i.e. $\lambda \in AP\sigma(a)$.
This proves (i), and assertion (ii) is an immediate consequence since $\sigma_r(a) = \overline{\sigma_\ell(a^*)}$ and $AP\sigma_r(a) = \overline{AP\sigma(a^*)}$ (where the bar means complex conjugation). Finally, (iii) follows from (i) and (ii).□

It follows at once that $\sigma(a)$, $\sigma_\ell(a)$, and $\sigma_r(a)$ are independent of the unital C*-algebra which contain the a_j, and that the sets of left invertible, right invertible, and invertible elements are open in A^n ([109], cors 2.4 and 2.6).

The clue of all descriptions of the spectra of elementary operators lies in the connections between the joint spectra of n-tuples and of their left and right regular representations. For $a = (a_1,\ldots,a_n) \in A^n$ we denote by L_a, R_a the n-tuples $L_a = (L_{a_1},\ldots,L_{a_n})$ and $R_a = (R_{a_1},\ldots,R_{a_n}) \in L(A)^n$, resp. We need the following "twenty-one sets" lemma.

A.2.7. *Let* A *be a unital C*-algebra and* $a \in A^n$. *Then*

$$\begin{array}{ccccccccc}
P\sigma_r(R_a) & = & AP\sigma_r(R_a) & = & \sigma_r(R_a) & \subset & \sigma(R_a) & \supset & \sigma_\ell(R_a) & = & AP\sigma(R_a) & \supset & P\sigma(R_a) \\
\cup & & \| & & \| & & \| & & \| & & \| & & \| \\
P\sigma(a) & \subset & AP\sigma(a) & = & \sigma_\ell(a) & \subset & \sigma(a) & \supset & \sigma_r(a) & = & AP\sigma_r(a) & \supset & P\sigma_r(a) \\
\| & & \| & & \| & & \| & & \| & & \| & & \cap \\
P\sigma(L_a) & \subset & AP\sigma(L_a) & = & \sigma_\ell(L_a) & \subset & \sigma(L_a) & \supset & \sigma_r(L_a) & = & AP\sigma_r(L_a) & = & P\sigma_r(L_a)
\end{array}$$

(Here $\subset$ means that strict inclusion can occur).

If, moreover, A *is a W*-algebra, then*

$$P\sigma(a) = \{\lambda \in \mathbb{C}^n \mid \bigwedge_j (1 - r_{\lambda_j - a_j}) \neq 0\}$$

$$= \{\lambda \in \mathbb{C}^n \mid \sum_{j=1}^n R_{\lambda_j - a_j} A \text{ is not } \sigma(A, A_*)\text{-dense}\}$$

and

$$P\sigma_r(a) = \{\lambda \in \mathbb{C}^n \mid \bigwedge_j (1 - \ell_{\lambda_j - a_j}) \neq 0\}$$

$$= \{\lambda \in \mathbb{C}^n \mid \sum_{j=1}^n L_{\lambda_j - a_j} A \text{ is not } \sigma(A, A_*)\text{-dense}\}$$

(where $\bigwedge_j p_j$ *denotes the infimum of projections* p_j*).*

Proof. Most of the inclusions follow directly from the Definition A.2.1 and A.2.2 (cf. also [72], 3.1), and A.2.6 changes the inclusions $AP\sigma(a) \subseteq AP\sigma(L_a) \subseteq \sigma_\ell(L_a) \subseteq \sigma_\ell(a)$ and $AP\sigma_r(a) \subseteq AP\sigma(R_a) \subseteq \sigma_\ell(R_a) \subseteq \sigma_r(a)$ into equalities. The remaining non-trivial inclusions are $\sigma_\ell(a) \subseteq P\sigma_r(R_a)$ and $\sigma_r(a) \subseteq P\sigma_r(L_a)$. For their proof observe that, for any $\lambda \in \mathbb{C}^n$, the left ideal $\sum_{j=1}^n A(\lambda_j - a_j)$ and the right ideal $\sum_{j=1}^n (\lambda_j - a_j)A$ are dense in A if and only if they are equal to A. Hence,

$$\sigma_\ell(a) = \{\lambda \in \mathbb{C}^n \mid \sum_{j=1}^n A(\lambda_j - a_j) \neq A\}$$

$$= \{\lambda \in \mathbb{C}^n \mid \sum_{j=1}^n A(\lambda_j - a_j) \text{ is not dense in } A\} = P\sigma_r(R_a)$$

and

$$\sigma_r(a) = \{\lambda \in \mathbb{C}^n \mid \sum_{j=1}^n (\lambda_j - a_j)A \neq A\}$$

$$= \{\lambda \in \mathbb{C}^n \mid \sum_{j=1}^n (\lambda_j - a_j)A \text{ is not dense in } A\} = P\sigma_r(L_a)$$

where we used A.2.2 again.

Counterexamples illustrating the possibly strict inclusions can be constructed easily.

In case A is a W*-algebra and $\lambda \in \mathbb{C}^n$ is fixed we put

$p_j = 1 - \ell_{\lambda_j - a_j}$ and $p = \bigwedge_j p_j$. Since $R^{-1}_{\lambda_j - a_j}(0) = Ap_j$ and $\bigcap_{j=1}^{n} Ap_j = Ap$ we find $\lambda \in P\sigma_r(a)$ iff $\bigcap_{j=1}^{n} R^{-1}_{\lambda_j - a_j}(0) \neq 0$ which in turn is equivalent to $p \neq 0$.

This yields the first assertion on the right point spectrum of a. For the second observe that $\sum_{j=1}^{n} (\lambda_j - a_j)A$ is $\sigma(A, A_*)$-dense in qA, where q is the supremum of all $\ell_{\lambda_j - a_j} = 1 - p_j$. Now, $q = 1 - p$ completes the argument.

The description of the point spectrum of a can be derived from an analogous argument, or by taking adjoints.□

The reader will find it easy to decide which *equalities* in A.2.7 are intrinsic to the C*-case, and which ones remain true for arbitrary unital Banach algebras. Therefore we refrain from restating A.2.7 for Banach algebras.

A.2.8. The concept of ultrapowers (cf. Appendix 1) is also useful in connection with spectral theory. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let E, F be Banach spaces with ultrapowers $\hat{E}_{\mathcal{U}}$, $\hat{F}_{\mathcal{U}}$ respectively.

(i) *The following conditions on* $T \in L(E,F)$ *are equivalent:*

(a) T *is bounded below.*

(b) $\hat{T}$ *is bounded below.*

(c) $\hat{T}$ *is injective.*

This is very well known, so we omit the proof. Its dual version, however, seems to be less well known (at least, we could not find a reference).

(ii) *The following conditions on* $T \in L(E,F)$ *are equivalent:*

(a) T *is open.*

(b) $\hat{T}$ *is open.*

(c) $\hat{T}$ *has dense range.*

Proof. (a) => (b) If T is open and $(y_k)_{k\in\mathbb{N}}$ is a bounded sequence in F, we find a bounded sequence $(x_k)_{k\in\mathbb{N}}$ in E with $Tx_k = y_k$ for each $k \in \mathbb{N}$. Thus $\hat{T}(x_k)_{\mathcal{U}} = (y_k)_{\mathcal{U}}$, and $\hat{T}$ is open by the Open Mapping Theorem.

(b) => (c) is trivial.

(c) => (a) If $\hat{T}$ has dense range then $\hat{T}' \in L(\hat{F}'_{\mathcal{U}}, \hat{E}'_{\mathcal{U}})$ is injective. By A.1.7, $\widehat{T'} \in L((F')\hat{}_{\mathcal{U}}, (E')\hat{}_{\mathcal{U}})$ is injective, hence $T' \in L(F',E')$ is bounded below by (i). Therefore, T has to be open by the Separation Theorem and the Open Mapping Theorem (cf. [17], 57.16). □

Remarks. 1. In playing around with the possible combinations of $'$ and $\hat{}$ one may add in (ii) the following further equivalent conditions:

(d) $\widehat{T'}$ *is bounded below,* (e) $\hat{T}'$ *is bounded below,* (f) $\widehat{T'}$ *is injective,* (g) $\hat{T}'$ *is injective.*

An even longer list is possible for (i).

2. Of course, there is no need to use ultrapowers to obtain (i) and (ii); any reduced power with respect to a filter that is finer than the Fréchet filter on $\mathbb{N}$ will suffice. Even the completeness of E and F is not essential (cf. [75], [76]).

As a consequence we obtain relations between joint spectra in A and in $\hat{A}_{\mathcal{U}}$.

A.2.9. *Let $\hat{E}_{\mathcal{U}}$ be the ultrapower of the Banach space E with respect to a free ultrafilter $\mathcal{U}$. If $T = (T_1,\dots,T_n) \in L(E)^n$ is any n-tuple of operators on E and $\hat{T} = (\hat{T}_1,\dots,\hat{T}_n) \in L(\hat{E}_{\mathcal{U}})^n$ is the n-tuple of the induced operators on $\hat{E}_{\mathcal{U}}$, then the following equations hold*

(i) $AP\sigma(T) = AP\sigma(\hat{T}) = P\sigma(\hat{T})$

(ii) $AP\sigma_r(T) = AP\sigma_r(\hat{T}) = P\sigma_r(\hat{T})$.

In particular, if A *is a unital Banach algebra, for any* $a \in A^n$ *we have*

$$AP\sigma(a) = AP\sigma(\hat{a}) = P\sigma(\hat{a}) \quad \textit{and} \quad AP\sigma_r(a) = AP\sigma_r(\hat{a}) = P\sigma_r(\hat{a}).$$

Proof. For (i) consider the auxiliary operator $\mathrm{Col}(T): E \to E^n$, $x \mapsto ((\lambda_1 - T_1)x, \ldots, (\lambda_n - T_n)x)$ where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. By A.2.2, $\mathrm{Col}(T)$ is injective iff $\lambda \notin P\sigma(T)$, and $\mathrm{Col}(T)$ is bounded below iff $\lambda \notin AP\sigma(T)$. Since $\mathrm{Col}(T)\hat{} = \mathrm{Col}(\hat{T}): \hat{E}_U \to \hat{E}_U{}^n$, (i) follows by A.2.8.(i).

For (ii) consider $\mathrm{Row}(T): E^n \to E$, $x \mapsto \sum_{j=1}^{n} (\lambda_j - T_j)x_j$, and note that $\mathrm{Row}(T)$ has dense range iff $\lambda \notin P\sigma_r(T)$, and $\mathrm{Row}(T)$ is surjective iff $\lambda \notin AP\sigma_r(T)$ (A.2.2). The assertion now follows by A.2.8.(ii).

Since $AP\sigma(a) = AP\sigma(L_a)$, $AP\sigma_r(a) = AP\sigma(R_a)$, $P\sigma(a) = P\sigma(L_a)$, and $P\sigma_r(a) = P\sigma(R_a)$ whenever $a \in A^n$ for a unital Banach algebra A (see above), the last assertion now is an immediate consequence.□

BIBLIOGRAPHY

1. C.A. Akemann, S. Wright, Compact and weakly compact derivations of C*-algebras, Pac. J. Math. 85 (1979), 253-259
2. C.A. Akemann, S. Wright, Compact actions on C*-algebras, Glasgow Math. J. 21 (1980), 143-149
3. E. Albrecht, J. Eschmeier, M. Neumann, Some topics in the theory of decomposable operators, in: Advances in invariant subspaces and other results of operator theory, Proc. 9th Int. Conf. on Operator Theory, Timişoara and Herculane, 1984, Birkhäuser Verlag, Basel-Boston-Stuttgart 1986, 15-34
4. E.M. Alfsen, Compact convex sets and boundary integrals, Ergeb. d. Math. 57, Springer Verlag, Berlin-Heidelberg-New York 1971
5. J. Anderson, C. Foiaş, Properties which normal operators share with normal derivations and related operators, Pac. J. Math. 61 (1975), 313-325
6. T. Ando, M.-D. Choi, Non-linear completely positive maps, in: Aspects of positivity in functional analysis, Proc. Conf. held on the occasion of H.H.Schaefer's 60th birthday, Tübingen, 24-28 June 1985, North Holland, Amsterdam 1986, 3-13
7. C. Apostol, L.A. Fialkow, Structural properties of elementary operators, Canad. J. Math. 38 (1986), 1485-1524
8. R.J. Archbold, Prime C*-algebras and antilattices, Proc. London Math. Soc. 24 (1972), 669-680
9. R.J. Archbold, On factorial states of operator algebras, J. Funct. Anal. 55 (1984), 25-38
10. R.J. Archbold, C.J.K. Batty, On factorial states of operator algebras.II, J. Operator Theory 13 (1985), 131-142
11. R.J. Archbold, C.J.K. Batty, On factorial states of operator algebras.III, preprint
12. W.B. Arveson, Subalgebras of C*-algebras, Acta Math. 123 (1969), 141-224

13. L. Asimov, A.J. Ellis, Convexity theory and its applications in functional analysis, Academic Press, London 1980

14. W.G. Badé, P.C. Curtis, Prime ideals and automatic continuity problems for Banach algebras. J. Funct. Anal. 29 (1978), 88-103

15. B.A. Barnes, G.J. Murphy, M.R.F. Smyth and T.T. West, Riesz and Fredholm theory in Banach algebras, Pitman Research Notes in Mathematics 67, Boston 1982

16. J.L. Bell, A.B. Slomson, Models and ultraproducts: An introduction, North Holland, Amsterdam 1974

17. S.K. Berberian, Lectures in functional analysis, Springer Verlag, New York 1974

18. F.F. Bonsall, Compact operators from an algebraic standpoint, Glasgow Math. J. 8 (1967), 41-49

19. F.F. Bonsall, Operators that act compactly on an algebra of operators, Bull. London Math. Soc. 1 (1969), 163-170

20. F.F. Bonsall, J. Duncan, Complete normed algebras, Springer Verlag, Berlin-Heidelberg-New York 1973

21. M. Breuer, Fredholm theories in von Neumann algebras.I, Math. Ann. 178 (1968), 243-254

22. M. Breuer, Fredholm theories in von Neumann algebras.II, Math. Ann. 180 (1969), 313-325

23. J.W. Calkin, Two-sided ideals and congruences in the ring of bounded operators in Hilbert space, Ann. Math. 42 (1941), 839-873

24. S.R. Caradus, W.E. Pfaffenberger, B. Yood, Calkin algebras and algebras of operators on Banach spaces, Marcel Dekker, New York 1974

25. A. Carrillo, C. Hernández, Spectra of constructs of a system of operators, Proc. Amer. Math. Soc. 91 (1984), 426-432

26. C.C. Chang, H.J. Keisler, Model theory, North Holland, Amsterdam 1977

27. S.J. Cho, V. Kaftal, A generalization of the Weyl-von Neumann theorem to semifinite von Neumann algebras, Indiana Univ. Math. J. 31 (1982), 57-63

28. M.-D. Choi, Positive linear maps on C*-algebras, Canad. J. Math. 24 (1972), 520-529

29. M.-D. Choi, Completely positive linear maps on complex matrices, Linear Alg. Appl. 10 (1975), 285-290

30. M.-D. Choi, Positive linear maps, Proc. Sym. Pure Math. 38 (1982), Amer. Math. Soc., Providence R.I., 583-590

31. M.-D. Choi, Ch. Davis, The spectral mapping theorem for joint approximate point spectrum, Bull. Amer. Math. Soc. 80 (1974), 317-321

32. M.-D. Choi, S.-K. Tsui, Tracial positive linear maps of C*-algebras, Proc. Amer. Math. Soc. 87 (1983), 57-61

33. C.-H. Chu, Prime ideals in C*-algebras, J. London Math. Soc. 7 (1973), 175-180

34. C.-H. Chu, On convexity theory and C*-algebras, Proc. London Math. Soc. 31 (1975), 257-288

35. F. Combes, Sur les faces d'une C*-algèbre, Bull. Sc. Math. 93 (1969), 37-62

36. R.E. Curto, Spectral permanence for joint spectra, Trans. Amer. Math. Soc. 270 (1982), 659-665

37. R.E. Curto, The spectra of elementary operators, Indiana Univ. Math. J. 32 (1983), 193-197

38. R.E. Curto, Connections between Harte and Taylor spectra, Rev. Roum. Math. Pures Appl. 31 (1986), 203-215

39. R.E. Curto, Applications of several complex variables to multiparameter spectral theory, preprint

40. R.E. Curto, L.A. Fialkow, The spectral picture of left and right multiplications, preprint

41. D. Dacunha-Castelle, Ultraproduits d'espaces de Banach, Séminaire Goulaouic-Schwartz, Paris 1971-1972, Exposé IX

42. D. Dacunha-Castelle, J.L. Krivine, Applications des ultra-produits à l'étude des espaces et des algèbres de Banach, Studia Math. 41 (1972), 315-334

43. H.G. Dales, Algebra and topology in Banach algebras, in: Proc. Conf. on Banach algebras and several complex variables, New Haven 1983, Contemporary Math. 32, Amer. Math. Soc., Providence R.I., 1984, 97-106

44. Ch. Davis, P. Rosenthal, Solving linear operator equations, Canad. J. Math. 26 (1974), 1384-1389

45. J. dePillis, Linear transformations which preserve hermitian and positive semidefinite operators, Pac. J. Math. 23 (1967), 129-137

46. J. Dixmier, Les C*-algèbres et leurs représentations, Gouthier-Villars, Paris 1969

47. P.G. Dixon, Semiprime Banach algebras, J. London Math. Soc. (2) 6 (1973), 676-678

48. R.G. Douglas, On the operator equation S*XT = X and related topics, Acta Sci. Math. (Szeged) 30 (1969), 19-32

49. N. Dunford, J.T. Schwartz, Linear operators Part I, Interscience, New York 1958

50. G.A. Elliott, L. Zsidó, Almost uniformly automorphism groups of operator algebras, J. Operator Theory 8 (1982), 227-277

51. M.R. Embry, M. Rosenblum, Spectra, tensor products, and linear operator equations, Pac. J. Math. 53 (1974), 95-107

52. D.E. Evans, J.T. Lewis, Dilations of irreversible evolutions in algebraic quantum theory, Comm. Dublin Inst. Adv. Studies Series A, no. 24 (1977)

53. L.A. Fialkow, A note on the operator $X \to AX - XB$, Trans. Amer. Math. Soc. 243 (1978), 147-168

54. L.A. Fialkow, A note on norm ideals and the operator $X \to AX - XB$, Israel J. Math. 32 (1979), 331-348

55. L.A. Fialkow, Elements of spectral theory for generalized derivations, J. Operator Theory 3 (1980), 89-113

56. L.A. Fialkow, Generalized derivations, in: Topics in modern operator theory, Proc. 5th Conf. on Operator Theory held in Timişoara and Herculane 1980, Birkhäuser Verlag, Stuttgart 1981, 95-103

57. L.A. Fialkow, A note on the range of the operator $X \to AX - X$ Illinois J. Math. 25 (1981), 112-124

58. L.A. Fialkow, Essential spectra of elementary operators, Trans. Amer. Math. Soc. 267 (1981), 157-174

59. L.A. Fialkow, Elements of spectral theory for generalized derivations II: The semi-Fredholm domain, Canad. J. Math. 33 (1981), 1205-1231

60. L.A. Fialkow, Spectral properties of elementary operators, Acta Sci. Math. 46 (1983), 269-282

61. L.A. Fialkow, Spectral properties of elementary operators.II, Trans. Amer. Math. Soc. 290 (1985), 415-429

62. L.A. Fialkow, R. Loebl, Elementary mappings into ideals of operators, Illinois J. Math. 28 (1984), 555-578

63. P.A. Fillmore, J.G. Stampfli, J.P. Williams, On the essential numerical range, the essential spectrum, and a problem of Halmos, Acta Sci. Math. 33 (1972), 179-192

64. C.K. Fong, A.R. Sourour, On the operator identity $\sum A_k X B_k = 0$, Canad. J. Math. 31 (1979), 845-857

65. J. Glimm, A Stone-Weierstrass theorem for C*-algebras, Ann. Math. 72 (1960), 216-244

66. J. Gravner, A note on elementary operators on the Calkin algebra, Proc. Amer. Math. Soc. 97 (1986), 79-86

67. M.D. Green, The lattice structure of C*-algebras and their duals, Math. Proc. Camb. Phil. Soc. 81 (1977), 245-248

68. U. Haagerup, A new proof of the equivalence of injectivity and hyperfiniteness for factors on a separable Hilbert space, J. Funct. Anal. 62 (1985), 160-201

69. H. Halpern, Spectra of operator equations and automorphism groups of von Neumann algebras, Math. Scand. 42 (1978), 135-149

70. H. Halpern, V. Kaftal, Compact operators in type III_λ and type III_0 factors, preprint

71. R.E. Harte, Spectral mapping theorems, Proc. R. Ir. Acad. 72A (1972), 89-107

72. R.E. Harte, Tensor products, multiplication operators and the spectral mapping theorem, Proc. R. Ir. Acad. 73A (1973), 285-302

73. R.E. Harte, A spectral mapping theorem for holomorphic functions, Math. Z. 154 (1977), 67-69

74. R.E. Harte, Invertibility, singularity and Joseph L. Taylor, Proc. R. Ir. Acad. 81A (1981), 71-79

75. R.E. Harte, Almost open mappings between normed spaces, Proc. Amer. Math. Soc. 90 (1984), 243-249

76. R.E. Harte, M. Mathieu, Enlargements of almost open mappings, Proc. Amer. Math. Soc. 96 (1986), 247-248

77. S. Heinrich, Ultraproducts in Banach space theory, J. Reine Angew. Math. 313 (1980), 72-104

78. S. Heinrich, Ultrapowers of locally convex spaces and applications,I, Math. Nachr. 118 (1984), 285-315

79. C.W. Henson, L.C. Moore,jr., The nonstandard theory of topological vector spaces, Trans. Amer. Math. Soc. 172 (1972), 405-435

80. D.A. Herrero, Approximation of Hilbert space operators, Vol. I, Pitman Research Notes in Mathematics 72, Pitman, Boston 1982

81. I.N. Herstein, Topics in ring theory, Univ. Chicago Lecture Notes, Chicago 1965

82. I.N. Herstein, Rings with involution, Chicago Lectures in Mathematics, Chicago - London 1976

83. R.D. Hill, Linear transformations which preserve hermitian matrices, Linear Alg. Appl. 6 (1973), 257-262

84. R.D. Hill, J.A. Poluikis, Completely positive and hermitian-preserving linear transformations, Linear Alg. Appl. 35 (1981), 1-10

85. M. Hladnik, On prespectrality of generalised derivations, Proc. Edinburgh Math. Soc. 104 A (1986), 93-106

86. M. Hladnik, When are generalized derivations spectral?,in: Operator Theory: Advances and Applications, Birkhäuser, Basel 1987, 215-226

87. Y. Ho, A note on derivations, Bull. Inst. Math. Acad. Sinica 5 (1977), 1-5

88. N. Jacobson, Structure of rings, Amer. Math. Soc. Colloq. Publ. 37, Providence R.I., 1964

89. A.A. Jafarian, A.R. Sourour, Spectrum preserving linear maps, J. Funct. Anal. 66 (1986), 255-261

90. B.E. Johnson, An introduction to the theory of centralizers, Proc. London Math. Soc. 14 (1964), 299-320

91. R.V. Kadison, Isometries of operator algebras, Ann. Math. 54 (1951), 325-338

92. R.V. Kadison, Order properties of bounded self-adjoint operators, Proc. Amer. Math. Soc. 2 (1951), 505-510

93. R.V. Kadison, A representation theory for commutative topological algebra, Mem. Amer. Math. Soc. 7 (1951)

94. V. Kaftal, On the theory of compact operators in von Neumann algebras I, Indiana Univ. Math. J. 26 (1977), 447-457

95. Y. Kato, A proof of Olsen's theorem, Math. Jap. 21 (1976), 127-128

96. K. Kraus, General state changes in quantum theory, Ann. Physics 64 (1971), 311-335

97. B. Kümmerer, Completely positive operators and their representations, preprint

98. B. Kümmerer, M. Mathieu, A characterization of completely positive elementary operators on C*-algebras, preprint

99. J. Lambek, Lectures on rings and modules, Blaisdell, Waltham-Toronto-London 1966

100. G. Lindblad, Completely positive maps and entropy inequalities, Comm. Math. Phys. 40 (1975), 147-151

101. G. Lindblad, On the generator of quantum dynamical semigroups, Comm. Math. Phys. 48 (1976), 119-130

102. G. Lumer and M. Rosenblum, Linear operator equations, Proc. Amer. Math. Soc. 10 (1959), 32-41

103. W.A.J. Luxemburg, A general theory of monads, in: Applications of model theory to algebra, analysis, and probability, Holt, Rinehart and Winston, New York 1969, 18-86

104. C.C. MacDuffee, The theory of matrices, Ergeb. d. Math. 2, Springer Verlag, Berlin 1933

105. B. Magajna, A system of operator equations, preprint

106. M. Marcus, B.N. Moyls, Linear transformations on algebras of matrices, Canad. J. Math. 11 (1959), 61-66

107. M. Marcus, R. Purves, Linear transformations on algebras of matrices: The invariance of the elementary symmetric functions, Canad. J. Math. 11 (1959), 383-396

108. M. Mathieu, On the spectrum of multiplications on von Neumann algebras, Semesterbericht Funktionalanalysis SS 1982, Tübingen 1982, 113-122

109. M. Mathieu, Spectral theory for multiplication operators on C*-algebras, Proc. R. Ir. Acad. 83A (1983), 231-249

110. M. Mathieu, A characterization of positive multiplications on C*-algebras, Math. Jap. 29 (1984), 375-382

111. M. Mathieu, Elementary operators on prime C*-algebras, submitted

112. N.H. McCoy, Prime ideals in general rings, Amer. J. Math. 71 (1949), 823-833

113. N.H. McCoy, The theory of rings, Macmillian, London 1970

114. C.L. Olsen, A structure theorem for polynomially compact operators, Amer. J. Math. 93 (1971), 686-698

115. C.L. Olsen, Index theory in von Neumann algebras, Mem. Amer. Math. Soc. 294, Vol.47 (1984)

116. G.K. Pedersen, C*-algebras and their automorphism groups, Academic Press, London 1979

117. E.C. Posner, Derivations in prime rings, Proc. Amer. Math. Soc. 8 (1957), 1093-1100

118. C.E. Rickart, General theory of Banach algebras, Van Nostrand, New York 1960

119. M. Rosenblum, On the operator equation $BX - XA = Q$, Duke Math. J. 23 (1956), 263-269

120. P. Rosenthal, On the equation $X = KXS$ and $AX = XK$, in: Spectral theory, Banach Center Publ. 8, Warsaw 1977, 389-391

121. S. Sakai, C*-algebras and W*-algebras, Springer Verlag, Berlin-Heidelberg-New York 1971

122. S. Sherman, Order in operator algebras, Amer. J. Math. 73 (1951), 227-232

123. A.M. Sinclair, Jordan homomorphisms and derivations on semi-simple Banach algebras, Proc. Amer. Math. Soc. 24 (1970), 209-214

124. Z. Słodkowski, An infinite family of joint spectra, Studia Math. 61 (1977), 239-255

125. M.K. Smith, Spectra of operator equations, Mich. Math. J. 23 (1976), 151-153

126. B. Stenström, Rings of quotients, Grundlehren d. math. Wiss. Bd. 217, Springer Verlag, Berlin-Heidelberg-New York 1975

127. M.C. Stéphanos, Sur une extension du calcul des substitution linéaires, J. Math. Pures Appl. V Vol. 6 (1900), 73-128

128. W.F. Stinespring, Positive functions on C*-algebras, Proc. Amer. Math. Soc. 6 (1955), 211-216

129. E. Størmer, Positive linear maps of operator algebras, Acta Math. 110 (1963), 233-278

130. I. Suciu, Eine natürliche Erweiterung der kommutativen Banach-algebren, Rev. Roum. Math. Pures Appl. 7 (1962), 483-491

131. I. Suciu, Bruchalgebren der Banachalgebren, Rev. Roum. Math. Pure Appl. 8 (1963), 313-316

132. J. Synnatzschke, Über eindimensionale Elemente linearer Algebren und einige Eigenschaften eines elementaren Operators, Math. Nachr. 123 (1985), 121-130

133. T. Takasaki, On some cases where n-positivity coincides with complete positivity, Sci. Rep. Niigata Univ. Ser. A No.18 (1982), 11-14

134. T. Takasaki, J. Tomiyama, Stinespring type theorems for various types of completely positive maps associated to operator algebras, Math. Jap. 27 (1982), 129-139

135. T. Takasaki, J. Tomiyama, On the geometry of positive maps in matrix algebras, Math. Z. 184 (1983), 101-108

136. M. Takesaki, Theory of operator algebras I, Springer Verlag, New York-Heidelberg-Berlin 1979

137. M. Takesaki, J. Tomiyama, Applications of fibre bundles to the certain class of C*-algebras, Tohôku Math. J. 13 (1961), 498-523

138. J.L. Taylor, A joint spectrum for several commuting operators, J. Funct. Anal. 6 (1970), 172-191

139. J. Tomiyama, On the difference of n-positivity and complete positivity in C*-algebras, J. Funct. Anal. 49 (1982), 1-9

140. J. Tomiyama, On the geometry of positive maps in matrix algebras,II, Linear Alg. Appl. 69 (1985), 169-177

141. S.-K. Tsui, Compact derivations on von Neumann algebras, Canad. Math. Bull. 24 (1981), 87-90

142. Y. Utumi, On quotients rings, Osaka Math. J. 8 (1956), 1-18

143. K. Vala, On compact sets of compact operators, Ann. Acad. Sci. Fenn. Ser. A I 351 (1964)

144. K. Vala, Sur les éléments compacts d'une algèbre normée, Ann. Acad. Sci. Fenn. Ser. A I 407 (1967)

145. R.E. Weber, On weak* continuous operators on $\mathcal{B}(H)$, Proc. Amer. Math. Soc. 83 (1981), 735-742

146. J.P. Williams, On the range of a derivation, Pac. J. Math. 38 (1971), 273-279

147. K. Ylinen, Compact and finite-dimensional elements of normed algebras, Ann. Acad. Sci. Fenn. Ser. A I 428 (1968)

148. K. Ylinen, Dual C*-algebras, weakly semi-completely continuous elements, and the extreme rays of the positive cone, Ann. Acad. Sci. Fenn. Ser. A I 599 (1975)

149. K. Ylinen, Weakly completely continuous elements of C*-algebras, Proc. Amer. Math. Soc. 52 (1975), 323-326

150. B. Yood, Closed prime ideals in topological rings, Proc. London Math. Soc. (3) 24 (1972), 307-323

151. W. Żelazko, On the divisors of zero of the group algebra, Fund. Math. 45 (1958), 99-102

152. W. Żelazko, Banach algebras, Elsevier, Amsterdam-London-New York 1973

INDEX OF SYMBOLS

$^{\circ}A$ 47

A_h 38, 51

A_+ 59

A^* 51

A_+^* 39

A_* 103

A' 66

A^{**} 37, 74

A_l^{**} 74

A_h^{**} 53

$\hat{A}$ 75, 120

$\hat{A}_U$ 114

$A \otimes {}^{\circ}A$ 47

$A + \mathbb{C}$ 116

$A(\Delta)$ 14

$A(U,B)$ 15

$A(K)$ 38

$A_o(\Omega)$ 40

$AP\sigma(a)$ 118

$AP\sigma_r(a)$ 118

(a) 11

$C(X)$ 87

$C(E)$ 72

$C(H)$ 72

c_o 14

c_M 12

conv 61

δ_b 52

$\delta_{a,b}$ 51

E' 113

E^{**} 72

$\hat{E}_F$ 13

$\hat{E}_U$ 108

$\hat{E}'_U$ 113

$(E')\hat{}_U$ 113

$(E_i)_U$ 107

$E \oplus F$ 114

$E\ell(A)$ 42

$E\ell_p(A)$ 61

$E\ell_{cp}(A)$ 61

$E\ell_{cp}(A;c)$ 66

$F(E)$ 15

$F(H)$ 86

$F(H_p)$ 79

$f_{(p)}$ 26

ϕ_p 69

$\mathrm{Hom}_A(I,A)$ 30

I_x 39

$I_{(p)}$ 26

I_{abel} 84

I_{lim} 85

I 19

I_c 19

$[(I,f)]$ 21

Inv(R) 120

(I_{min}, f_{min}) 25

$I_1 \wedge I_2$ 19

$I_1 \vee I_2$ 19

$(I_1,f_1) \wedge (I_2,f_2)$ 25

$(I_1,f_1) \vee (I_2,f_2)$ 25

ι 74

inf(a,b) 39

$K(A)$ 71

$K(E)$ 71

$K(H)$ 72

L_a 122

$L(E)$ 8

$L(E,F)$ 8

$L(H)$ 8

$L(A)_p$ 61

$L(A)_{cp}$ 61

$LM(I)$ 31

$L^1[0,1]$ 72

$L^1(\mathbb{R}_+)$ 14

$L^1(G)$ 14

ℓ_x 104

$\ell(S)$ 38

$\ell^\infty(I,E_i)$ 107

$\ell^\infty(E)$ 111

$\lim_U \|x_i\|$ 107

$^\circ M$ 47

M 11

$M(A)$ 42

$M(A)_p$ 61

$M_{a,b}$ 10

$M(A)$ 36

$M_n(A)$ 59, 116

M_n 9, 60

$\mathfrak{M}$ 19

$\mathfrak{M}_c$ 19

$\mathfrak{M}_a$ 23

$\mathfrak{M}^c$ 23

$m(A)$ 83

N 28

$n_U(E)$ 108

n_U 107

$P(X)$ 109

$P\sigma(a)$ 118

$P\sigma_r(a)$ 118

π_p 79

π_ϕ 40

$\pi(A)''$ 74

$\pi(A)_1''$ 74

$\bigoplus_{t\in A}\pi_t$ 75

π 74

πS 69, 74

$\bigwedge_j p_j$ 123

Q 40

$Q(A)$ 21

$Q_a(A)$ 23

$Q_c(A)$ 23

Q_A 28

$\tilde{Q}_A$ 29

R_a 122

r_x 104

$r(x)$ 50

ρ_r 37

ρ_ℓ 37

S 42

$S_|$ 95

$S_/$ 95

$S(A)$ 33, 52

$soc(A)$ 77

$\sum^{\oplus}_{t\in A} L(H_t)$ 75

$\sum^{\oplus}_{t\in A_0} K(H_t)$ 76

sup q 24

$\sigma(a)$ 119

$\sigma_\ell(a)$ 119

$\sigma_r(a)$ 119

$\sigma_B(a)$ 121

$\delta(\phi)$ 106

δ_ℓ 106

δ_r 106

T' 113, 119

$\hat{T}'$ 113

$\hat{T}$ 113

$\hat{T}'$ 113

T^{**} 72

$T^{(n)}$ 59

τ 5, 63

θ 47

$U \times V$ 109

$V(a)$ 51

$W(E)$ 72

$\mathfrak{X}$ 108

$\xi \otimes \eta$ 8

Y^c 110

$Z(A)$ 14

Lebenslauf

Am 11. Mai 1957 wurde ich in Heidelberg am Neckar geboren. In den folgenden Jahren reisten mein Bruder Michael und ich mit unseren Eltern - Vater Georges Mathieu, von Beruf Kapellmeister und Komponist, Mutter Ingeborg Mathieu, geborene Durchbach, von Beruf Opernsängerin - während ihrer Tourneen durch ganz Deutschland und Teile der Schweiz. Im Jahr 1962 pachtete mein Vater eine Gaststätte in Rosenheim/Obb., wo ich dann 1963 eingeschult wurde. Nach der Volksschulzeit in Rosenheim, Riedering und Rott am Inn wechselte ich 1967 in das Luitpold-Gymnasium in Wasserburg am Inn. Im Juni 1976 legte ich dort die Reifeprüfung ab.

Im Oktober desselben Jahres begann ich mein Studium an der Eberhard-Karls-Universität Tübingen - Diplom-Mathematik mit Nebenfach Physik -, ab dem vierten Semester unter Förderung der Studienstiftung des deutschen Volkes. Im Herbst 1978 absolvierte ich das Vordiplom. Einen Großteil meiner Studien nach dem Vordiplom widmete ich der Funktionalanalysis, der unendlichdimensionalen Komplexen Analysis und der Theorie der Operatoralgebren. Während meines Hauptstudiums war ich mehrfach als wissenschaftliche Hilfskraft am Mathematischen Institut tätig. Meine Diplomarbeit schrieb ich über das Thema "Positive Multiplikatoren auf C*-Algebren". Im September 1982 legte ich die Diplomprüfung in Mathematik ab, im Dezember 1982 folgte die Nebenfachprüfung in Theoretischer Physik.

Ab April 1984 erhielt ich ein Promotionsstipendium der Studienstiftung des deutschen Volkes, im Rahmen dessen ich mich von Oktober bis Dezember 1984 am Trinity College in Dublin aufhielt und dort auch eine Vorlesung über Funktionalanalysis für Studenten im dritten Studienjahr las. Vom 26.8. bis 4.9. 1986 nahm ich an der 10. Internationalen Konferenz über Operatorentheorie in Bukarest, Rumänien teil. Seit November 1985 vertrete ich eine Assistentenstelle am Lehrstuhl für mathematische Methoden der Naturwissenschaften. Im Dezember 1985

besuchte ich für eine Woche das Mathematische Institut der Universität Ljubljana, Jugoslawien, und im März 1986 nahm ich am Joint Meeting of the London Mathematical Society and the Irish Mathematical Society on Operator Theory and C*-algebras in Dublin teil. Im August und September 1986 folgte ich einer Einladung an die University of California at Santa Barbara. Für diesen Aufenthalt erhielt ich ein Stipendium des DAAD.

Seit Beginn meines Studiums wohne ich in Tübingen-Unterjesingen. Im Februar 1979 heiratete ich dort meine Frau Susanne, geborene Süßer. Am 20.Januar 1983 wurde unsere Tochter Janine geboren, am 14. Mai 1986 unser Sohn Jan.

Meine akademischen Lehrer an der Eberhard-Karls-Universität Tübingen waren

im Fach Mathematik die Herren Professoren und Doktoren
B. Blackadar, H. Fischer, W. Grölz, H. de Groote, H. Hähl, H. Heyer, W. Kaup, W. Knapp, R. Nagel, D. Petz, H.H. Schaefer, U. Schlotterbeck, P. Schmid, A. Schönhage, H. Upmeier und M. Wolff,

im Fach Physik die Herren Professoren und Doktoren
M. Baumann, A. Fässler, F. Gönnenwein, E. Kasper, H. Pfister und K. Wildermuth,

im Fach Geschichte der Naturwissenschaften die Herren Professoren und Doktoren
Krieger, F. Rex und M. Schramm.

Wissenschaftliche Arbeiten

On the origin of the notion ergodic theory,
Tübingen 1981, unpublished

On the spectrum of multiplications on von Neumann algebras,
Semesterbericht Funktionalanalysis SS 82, Tübingen 1982, 113-122

Spectral theory for multiplication operators on C*-algebras,
Proc. R. Ir. Acad. 83A (1983), 231-249

A characterization of positive multiplications on C*-algebras,
Math. Jap. 29 (1984), 375-382

A remark on elementary operators,
Semesterbericht Funktionalanalysis WS 83/84, Tübingen 1984, 239-247

On weakly compact elementary operators,
Semesterbericht Funktionalanalysis WS 84/85, Tübingen 1985, 177-185

Enlargements of almost open mappings,
Proc. Amer. Math. Soc. 96 (1986), 247-248 (with Robin Harte)

On approximate point spectra of elementary operators,
Semesterbericht Funktionalanalysis WS 85/86, Tübingen 1986,
105-116

A short geometric proof of Archbold's characterization of prime
C*-algebras as antilattices,
Semesterbericht Funktionalanalysis WS 85/86, Tübingen 1986,
101-104

Elementary operators on prime C*-algebras,
submitted

A characterization of completely positive elementary operators
on C*-algebras,
preprint (with Burkhard Kümmerer)